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**N° 1926**
Mai 1993

PROGRAMME 5

Traitement du signal,
automatique et
productique

Rapport de recherche

1993

# On an Investment-Consumption model with transaction costs

# Gestion de portefeuille avec coûts de transaction

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**Abstract :** This paper considers the optimal consumption and investment policy for an investor who has available one bank account paying a fixed interest rate $r$ and $n$ risky assets whose prices are log-normal diffusions. We suppose that transactions between the assets incur a cost proportional to the size of the transaction. The problem is to maximize the total utility of consumption. Dynamic Programming leads to a Variational Inequality for the value function. Existence and uniqueness of a viscosity solution is proved. The Variational Inequality is solved, by using a numerical algorithm based on policies iterations and multigrid methods. Numerical results are displayed for $n = 1$ and $n = 2$.

**Résumé :** On étudie la politique d'investissement et de consommation optimale d'un agent possédant un compte en banque à intérêt fixe et $n$ comptes en actions dont les prix sont modélisés par des processus de diffusion log-normaux. On suppose que chaque transaction entre les comptes s'accompagne d'un coût proportionnel au montant de la transaction. Le problème est de maximiser une fonction d'utilité de la consommation. On démontre que la fonction valeur est l'unique solution de viscosité d'une inéquation variationnelle. Cette inéquation variationnelle est ensuite résolue en utilisant une méthode numérique basée sur l'algorithme de Howard (itérations sur les politiques) et la méthode multigrille. Des résultats numériques sont présentés pour $n = 1$ et $n = 2$.

**Key-words :** Portfolio selection, transaction costs, viscosity solution, variational inequality, multigrid methods.

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# 1 Introduction

This paper concerns the theoretical and numerical study of a portfolio selection problem. Consider an investor who has available one riskless bank account paying a fixed rate of interest $r$ and $n$ risky assets modeled by log-normal diffusions with expected rates of return $\alpha_i > r$ and rates of return variation $\sigma_i^2$. The investor consumes at rate $c(t)$ from the bank account. Any movement of money between the assets incurs a transaction cost proportional to the size of the transaction, paid from the bank account. The investor is allowed to have a short position in one of the holdings, but his position vector must remain in the closed solvency region $\mathcal{S}$ defined as the set of positions for which the net wealth is nonnegative. The investor's objective is to maximize over an infinite horizon the expected discounted utility of consumption with a HARA (Hyperbolic Absolute Risk Aversion) type utility function.

This problem was formulated by Magill and Constantinides [20] for $n = 1$ who conjectured that the no-transaction region is a cone in the two-dimensional space of position vectors. This fact was proved in a discrete-time setting by Constantinides [7] who proposed an approximate solution based on making some assumptions on the consumption process. Davis and Norman proved, in continuous time, without this restriction, that the optimal strategy confines indeed the investor's portfolio to a wedge-shaped region in the portfolio plane [9]. An analysis of the optimal strategy together with regularity results for the value function can be found in Fleming and Soner [12, chapter 8.7] and Shreve and Soner [26]. Taksar, Klass and Assaf [29] consider a model without consumption and study the problem of maximizing the long-run average growth of wealth. A deterministic model is solved by Shreve, Soner and Xu [27] with a general utility function which is not necessarily a HARA type function. A stochastic model driven by a finite state Markov chain rather than a Brownian motion and with a general but bounded utility function has been investigated in Zariphopolou [30]. She supposes that the amount of money allocated in the assets must remain non-negative and shows that the value function is the unique constrained viscosity solution of a system of variational inequalities with gradient constraints. Fitzpatrick and Fleming [11] study numerical methods for the optimal investment-consumption model with possible borrowing. They examine a Markov chain discretization of the original continuous problem similar to Kushner's numerical schemes [17]. The convergence arguments rely on viscosity solution techniques.

We consider here an extension of Davis and Norman's model to the case where more than one risky asset is allowed. We restrict to power utility functions of the form $\frac{c^\gamma}{\gamma}$ with $0 < \gamma < 1$. The mathematical formulation of the problem is given in section 2. In section 3, we prove that the value function is the unique viscosity solution of the associated variational inequality. Since the utility and the drift functions are not bounded, the uniqueness does not derive from classical results. We are then concerned with the numerical solution of the variational inequality. First an adequate change of variables is performed (see section 4) which reduces the dimension of the problem and simplifies the numerical study. Then, the variational inequality is discretized by finite difference schemes and solved by using algorithms based on the "Howard algorithm" (policy iteration) and the multigrid method (see section 5). Numerical results for the value function and the optimal policy in the case of one bank account and one or two risky asset(s) are given in section 6. Finally, in section 7, a theoretical study of the optimal strategy is done by using properties of the variational inequality; this analysis corroborates the numerical results.

## 2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ a given filtration. We denote by $s_0(t)$ (resp. $s_i(t)$ for $i = 1 \dots n$) the amount of money in the bank account (resp. in the i-th risky asset) at time $t$ and refer as $s(t) = (s_i(t))_{i=0,\dots,n}$ the investor position at time $t$. We suppose that the evolution equations of the investor holdings are

$$\begin{cases} ds_0(t) = (rs_0(t) - c(t))dt + \sum_{i=1}^{n}(-(1+\lambda_i)d\mathcal{L}_i(t) + (1-\mu_i)d\mathcal{M}_i(t)), \\ ds_i(t) = \alpha_i s_i(t)dt + \sigma_i s_i(t)dW_i(t) + d\mathcal{L}_i(t) - d\mathcal{M}_i(t), \quad i = 1, \dots, n, \end{cases} \tag{1}$$

with initial values

$$s_i(0^-) = x_i, \quad i = 0, \dots, n \tag{2}$$

where $W_i(t)$, $i = 1, \dots, n$, are independent Wiener processes, $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$ represent cumulative purchase and sale of stock $i$ on $[0, t]$ respectively and $s(t^-)$ denotes the left hand limit of the process $s$ at time $t$. The coefficients $\lambda_i$ and $\mu_i$ represent the proportional transaction costs.

A policy for investment and consumption is a set $(c(t), (\mathcal{L}_i(t), \mathcal{M}_i(t))_{i=1,\dots,n})$ of adapted processes such that

- $c(t,\omega) \geq 0, \quad \int_0^t c(s,\omega)ds < \infty$, for a.e. $(t,\omega)$,
- $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$ are right-continuous, non-decreasing and $\mathcal{L}_i(0^-) = \mathcal{M}_i(0^-) = 0$.

The process $s(t)$ is thus right continuous with left-hand limit and equations (1) and (2) are equivalent to :

$$\begin{cases} s_0(t) &= x_0 + \int_0^t (rs_0(\theta) - c(\theta))d\theta + \sum_{i=1}^{n} (-(1+\lambda_i)\mathcal{L}_i(t) + (1-\mu_i)\mathcal{M}_i(t)), \\ s_i(t) &= x_i + \int_0^t \alpha_i s_i(\theta)d\theta + \int_0^t \sigma_i s_i(\theta)dW_i(\theta) + \mathcal{L}_i(t) - \mathcal{M}_i(t), \quad i = 1, \dots, n, \end{cases}$$

for $t \geq 0$.

We define the solvency region as :

$$\mathcal{S} = \{x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}, \quad \mathcal{W}(x) \geq 0\}$$

where

$$\mathcal{W}(x) = x_0 + \sum_{i=1}^{n} \min((1-\mu_i)x_i, (1+\lambda_i)x_i) \tag{3}$$

represents the net wealth, that is the amount of money in the bank account resulting from setting the risky assets to zero.

Suppose that the investor is given an initial endowment $x$ in $\mathcal{S}$. A policy is admissible if the bankruptcy time $\bar{\tau}$ defined as

$$\bar{\tau} = \inf \{t \geq 0, s(t) \notin \mathcal{S}\} \tag{4}$$

is infinite. We denote by $\mathcal{U}(x)$ the set of admissible policies. The investor's objective is to maximize over all policies $\mathcal{P}$ in $\mathcal{U}(x)$ the discounted utility of consumption

$$J_x(\mathcal{P}) = E_x \int_0^\infty e^{-\delta t} u(c(t)) dt \tag{5}$$

where $E_x$ denotes expectation given the initial endowment $x$, $\delta$ is a positive discount factor and $u(c)$ is a utility function defined by

$$u(c) = \frac{c^\gamma}{\gamma}, \quad 0 < \gamma < 1. \tag{6}$$

We define the value function $V$ as

$$V(x) = \sup_{\mathcal{P} \in \mathcal{U}(x)} J_x(\mathcal{P}). \tag{7}$$

**Remark 2.1** When the process $s(t)$ reaches the boundary $\partial\mathcal{S}$ at time $t$, i.e. $s(t^-) \in \partial\mathcal{S}$, the only admissible policy is to jump immediately to the origin and remain there (see [27]) with a null consumption. Consequently, if the initial endowment $x$ is on the boundary, then $V(x) = 0$. ■

**Remark 2.2** Let $\tau$ denote the exit time of the interior of $\mathcal{S}$, defined as

$$\tau = \inf \left\{ t \geq 0, s(t) \notin \overset{\circ}{\mathcal{S}} \right\}. \tag{8}$$

We have for all admissible policies $\mathcal{P}$

$$J_x(\mathcal{P}) = E_x \int_0^\tau e^{-\delta t} u(c(t)) dt. \tag{9}$$

On the other hand, for any policy $\mathcal{P}$, we can construct an admissible policy which coincides with $\mathcal{P}$ until time $\tau$ (it is such that the process $s(t)$ jumps to the origin at time $\tau$). The value function can thus be rewritten as

$$V(x) = \sup_{\mathcal{P} \in \mathcal{U}} E_x \int_0^\tau e^{-\delta t} u(c(t)) dt \tag{10}$$

where $\mathcal{U}$ is the set of all policies. ■

We make the assumptions

$$\delta > \gamma \left( r + \frac{1}{2(1-\gamma)} \sum_{i=1}^n (\frac{\alpha_i - r}{\sigma_i})^2 \right), \tag{A.1}$$

and

$$0 \leq \mu_i < 1, \quad \lambda_i \geq 0, \quad \lambda_i + \mu_i > 0, \quad \forall i = 1, \ldots, n. \tag{A.2}$$

**Remark 2.3** When the transaction costs are equal to zero (Merton's problem), the value function $V$ is finite iff assumption (A.1) is satisfied (see [9] for $n = 1$ and section 7 below). ■

# 3 The Variational Inequality

We state the main theorem.

**Theorem 3.1** *Under assumptions (A.1) and (A.2),*

*(i) the value function $V$ defined in (7) or (10) is $\gamma$-Hölder continuous and concave in $\mathcal{S}$ and non decreasing with respect to $x_i$ for $i = 0, \ldots n$.*

*(ii) $V$ is the unique viscosity solution of the variational inequality (VI) :*

$$\max\left\{AV + u^*(\frac{\partial V}{\partial x_0}), \max_{1\leq i\leq n} L_i V, \max_{1\leq i\leq n} M_i V\right\} = 0 \qquad in\ \overset{\circ}{\mathcal{S}} \tag{11}$$

$$V/\partial\mathcal{S} = 0 \tag{12}$$

*where*

$$AV = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2 \frac{\partial^2 V}{\partial {x_i}^2} + \sum_{i=1}^{n}\alpha_i x_i \frac{\partial V}{\partial x_i} + r x_0 \frac{\partial V}{\partial x_0} - \delta V, \tag{13}$$

$$L_i V = -(1+\lambda_i)\frac{\partial V}{\partial x_0} + \frac{\partial V}{\partial x_i}, \tag{14}$$

$$M_i V = (1-\mu_i)\frac{\partial V}{\partial x_0} - \frac{\partial V}{\partial x_i}, \tag{15}$$

*and $u^*$ is the convex Legendre transform of $u$ defined by*

$$\begin{aligned} u^*(\nu) &= \max_{c\geq 0}(-c\nu + \frac{c^\gamma}{\gamma}) \\ &= (\frac{1}{\gamma} - 1)\nu^{\frac{\gamma}{\gamma-1}}. \end{aligned} \tag{16}$$

■

The solvency region $\mathcal{S}$ is divided as follows :

$$B_i = \{x \in \mathcal{S},\ L_i V(x) = 0\}, \tag{17}$$

$$S_i = \{x \in \mathcal{S},\ M_i V(x) = 0\}, \tag{18}$$

$$NT_i = \mathcal{S} \setminus (B_i \cup S_i), \tag{19}$$

$$NT = \bigcap_{i=1}^{n} NT_i. \tag{20}$$

$NT$ is the no transaction region. Outside $NT$, an instantaneous transaction brings the position to the boundary of $NT$ : buy stock $i$ in $B_i$, sell stock $i$ in $S_i$. After the initial transaction, the agent position remains in

$$\overline{NT} = \{x \in \mathcal{S}, AV + u^*\left(\frac{\partial V}{\partial x_0}\right) = 0\},$$

and further transactions occur only at the boundary (see [9]).

We shall first recall the definition of viscosity solutions, then prove points $(i)$ and $(ii)$ of Theorem 3.1 in sections 3.2 and 3.3 respectively.

## 3.1 Viscosity solutions of non linear elliptic equations

Consider fully nonlinear elliptic equations of the form

$$F(D^2v, Dv, v, x) = 0 \quad \text{in} \quad \mathcal{O} \tag{21}$$

where $F$ is a given continuous function in $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{O}$, $S^N$ is the space of symmetric $N \times N$ matrices, $\mathcal{O}$ is an open domain of $\mathbb{R}^N$, and the ellipticity of (21) is expressed by

$$F(A, p, v, x) \geq F(B, p, v, x) \quad \text{if } A \geq B, \;\; A, B \in S^N,\; p \in \mathbb{R}^N,\; v \in \mathbb{R},\; x \in \mathcal{O}. \tag{22}$$

A special case of (21) is given by

$$F(X, p, v, x) = \max_{\nu \in U}\{\sum_{i,j=1}^{N} a_{ij}(x,\nu)X_{ij} + \sum_{i=1}^{N} b_i(x,\nu)p_i - c(x,\nu)v + f(x,\nu)\} \tag{23}$$

where (22) is satisfied when the matrix $(a_{ij}(x,\nu))_{i,j}$ is symmetric non negative in $\mathcal{O} \times U$.
The variational inequality (11) is of this form.

**Definition 3.1** *Let $v \in C(\mathcal{O})$. Then $v$ is a viscosity solution of (21) if the following relations hold :*

$$F(X, p, v(x), x) \geq 0, \;\; \forall (p, X) \in J^{2,+}v(x), \;\; \forall x \in \mathcal{O} \tag{24}$$

$$F(X, p, v(x), x) \leq 0, \;\; \forall (p, X) \in J^{2,-}v(x), \;\; \forall x \in \mathcal{O} \tag{25}$$

*where $J^{2,+}$ and $J^{2,-}$ are the second order "superjets" defined by :*

$$J^{2,+}v(x) = \{(p, X) \in \mathbb{R}^N \times S^N,$$
$$\limsup_{\substack{y \to x \\ y \in \mathcal{O}}}\{[v(y) - v(x) - (p, y - x) - \frac{1}{2}(X(y - x), y - x)]|y - x|^{-2}\} \leq 0\}$$

*and*

$$J^{2,-}v(x) = \{(p, X) \in \mathbb{R}^N \times S^N,$$
$$\liminf_{\substack{y \to x \\ y \in \mathcal{O}}}\{[v(y) - v(x) - (p, y - x) - \frac{1}{2}(X(y - x), y - x)]|y - x|^{-2}\} \geq 0\}.$$

A viscosity subsolution (resp. supersolution) of (21) is similarly defined as an upper semicontinuous function satisfying (24) (resp. a lower semicontinuous function satisfying(25)) (see [8]).

## 3.2 Properties of the value function

**Lemma 3.1** *There exists a positive constant, a, such that the function*

$$\varphi(x) = a\left(x_0 + \sum_{i=1}^{n} \min((1-\mu_i)x_i, (1+\lambda_i)x_i)\right)^\gamma$$

*is a viscosity supersolution of equation (11) with $\varphi = 0$ on $\partial\mathcal{S}$.*

**Proof** : Denote $\nu = (\nu_1, \ldots, \nu_n)$ where $\nu_i$ is either equal to $\mu_i$ or $-\lambda_i$. The function $\varphi$ can be rewritten as

$$\varphi = \min_\nu \varphi_\nu$$

where

$$\varphi_\nu(x) = a\mathcal{W}_\nu(x)^\gamma \tag{26}$$

and

$$\mathcal{W}_\nu(x) = x_0 + \sum_{i=1}^{n}(1-\nu_i)x_i. \tag{27}$$

We have

$$L_i\varphi_\nu(x) = -(\lambda_i + \nu_i)a\gamma\mathcal{W}_\nu(x)^{\gamma-1} \leq 0, \tag{28}$$

$$M_i\varphi_\nu(x) = (\mu_i - \nu_i)a\gamma\mathcal{W}_\nu(x)^{\gamma-1} \leq 0, \tag{29}$$

and

$$A\varphi_\nu(x) = a\gamma\mathcal{W}_\nu(x)^{\gamma-2}G(x)$$

with

$$\begin{aligned} G(x) &= \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2(1-\nu_i)^2(\gamma-1) \\ &+ \mathcal{W}_\nu(x)\left(\sum_{i=1}^{n}(\alpha_i - r)x_i(1-\nu_i)\right) \\ &+ \mathcal{W}_\nu(x)^2\left(r - \frac{\delta}{\gamma}\right). \end{aligned}$$

Assumption (A.1) implies that there exists $\eta > 0$ such that

$$\frac{\delta}{\gamma} - r - \eta \geq \frac{1}{2(1-\gamma)}\sum_{i=1}^{n}\left(\frac{\alpha_i - r}{\sigma_i}\right)^2$$

which leads to

$$G(x) \leq -\eta\mathcal{W}_\nu(x)^2$$

and

$$A\varphi_\nu(x) \leq -a\gamma\eta\mathcal{W}_\nu(x)^\gamma = -\gamma\eta\varphi_\nu(x). \tag{30}$$

Moreover,

$$u^*(\frac{\partial \varphi_\nu}{\partial x_0}(x)) = u^*(a\gamma \mathcal{W}_\nu(x)^{\gamma-1}) = (\frac{1}{\gamma}-1)(a\gamma)^{\frac{\gamma}{\gamma-1}}\mathcal{W}_\nu(x)^\gamma = (1-\gamma)(a\gamma)^{\frac{1}{\gamma-1}}\varphi_\nu(x).$$

The constant $a$ can now be chosen such that

$$A\varphi_\nu + u^*(\frac{\partial \varphi_\nu}{\partial x_0}) \leq 0 \quad \text{in} \quad \overset{\circ}{\mathcal{S}}. \tag{31}$$

Since (28), (29), (31) hold, $\varphi_\nu$ is a classical supersolution of (11) (continuous in $\mathcal{S}$ and twice continuously differentiable in $\overset{\circ}{\mathcal{S}}$). Hence, $\varphi$ is a viscosity supersolution of (11) as the minimum of continuous supersolutions, and clearly vanishes on $\partial\mathcal{S}$. ■

**Proposition 3.1** *$V$ is uniformly $\gamma$-Hölder continuous in $\mathcal{S}$, that is*

$$\exists C > 0, \quad |V(x) - V(x')| \leq C\|x - x'\|^\gamma, \qquad \forall x, x' \in \mathcal{S}. \tag{32}$$

**Proof** : We consider two initial positions $x$ and $x'$, and denote by $\tau$ (resp. $\tau'$) the first exit time of $\overset{\circ}{\mathcal{S}}$ of the process $s(t)$ (resp. $s'(t)$) defined by (1) and $s(0^-) = x$ (resp. $s'(0^-) = x'$). We have

$$\begin{aligned} V(x) - V(x') &= \sup_{\mathcal{P}\in\mathcal{U}} E\int_0^\tau e^{-\delta t}u(c(t))dt - \sup_{\mathcal{P}\in\mathcal{U}} E\int_0^{\tau'} e^{-\delta t}u(c(t))dt \\ &\leq \sup_{\mathcal{P}\in\mathcal{U}} E\left(\int_0^\tau e^{-\delta t}u(c(t))dt - \int_0^{\tau'} e^{-\delta t}u(c(t))dt\right) \\ &\leq \sup_{\mathcal{P}\in\mathcal{U}} E\int_{\tau\wedge\tau'}^\tau e^{-\delta t}u(c(t))dt. \end{aligned}$$

The function $\varphi_\nu$ defined by (26) has $\mathcal{C}^2$-regularity and is a supersolution of (11). We apply Ito's formula for càdlàg processes (see [24]) to $e^{-\delta t}\varphi_\nu(s(t))$. For any stopping time $\theta$, the process

$$\begin{aligned} & e^{-\delta(\theta\wedge\tau)}\varphi_\nu(s(\theta\wedge\tau)) \\ & \quad - \int_0^{\theta\wedge\tau} e^{-\delta t}\left\{A^{c(t)}\varphi_\nu(s(t))dt + \sum_{i=1}^n [L_i\varphi_\nu(s(t))d\mathcal{L}_i{}^c(t) + M_i\varphi_\nu(s(t))d\mathcal{M}_i{}^c(t)]\right\} \\ & \quad - \sum_{0\leq t\leq\theta\wedge\tau} e^{-\delta t}[\varphi_\nu(s(t)) - \varphi_\nu(s(t^-))] \end{aligned}$$

is a martingale. We denote by $A^c$ the operator $A - c\frac{\partial}{\partial x_0}$ and by $\mathcal{L}_i{}^c(t)$ and $\mathcal{M}_i{}^c(t)$ the continuous parts of $\mathcal{L}_i(t)$ and $\mathcal{M}_i(t)$.

Since $s(t)$ has a jump only when $\mathcal{L}_i(t)$ or $\mathcal{M}_i(t)$ is discontinuous, we have

$$\mathcal{W}_\nu(s(t)) = \mathcal{W}_\nu(s(t^-)) - \sum_{i=1}^n [(\lambda_i + \nu_i)(\mathcal{L}_i(t) - \mathcal{L}_i(t^-)) + (\mu_i - \nu_i)(\mathcal{M}_i(t) - \mathcal{M}_i(t^-))] \leq \mathcal{W}_\nu(s(t^-)).$$

Hence,

$$\varphi_\nu(s(t)) \leq \varphi_\nu(s(t^-)).$$

In addition, $\mathcal{L}_i{}^c$ and $\mathcal{M}_i{}^c$ are non decreasing. Consequently

$$M_t^\nu = \varphi_\nu(s(t \wedge \tau))e^{-\delta(t\wedge\tau)} + \int_0^{t\wedge\tau} u(c(\theta))e^{-\delta\theta}d\theta$$

is a supermartingale, as well as the process $\min_\nu M_t^\nu$. Therefore

$$E\int_{\tau\wedge\tau'}^{\tau} e^{-\delta t}u(c(t))dt \leq E(e^{-\delta(\tau\wedge\tau')}\varphi(s(\tau\wedge\tau')) - e^{-\delta\tau}\varphi(s(\tau)))$$

and as $\varphi$ vanishes on $\partial\mathcal{S}$, we have

$$V(x) - V(x') \leq \sup_{\mathcal{P}\in\mathcal{U}} E(e^{-\delta\tau'}(\varphi(s(\tau')) - \varphi(s'(\tau')))1_{\tau'<\tau})$$

where $1_A$ denotes the characteristic function of the set $A$.

Let us fix for instance $\|x\| = \sup_{i=0,\ldots,n} |x_i|$. The function $\varphi$ is $\gamma$-Hölder continuous, that is

$$|\varphi(x) - \varphi(x')| \leq C\|x - x'\|^\gamma$$

for some positive constant $C$. We thus get

$$V(x) - V(x') \leq C \sup_{\mathcal{P}\in\mathcal{U}} E(e^{-\delta\tau'}\|(s(\tau') - s'(\tau')\|^\gamma 1_{\tau'<\tau}). \tag{33}$$

The process $\Sigma(t) = s(t) - s'(t)$ is a diffusion process with generator $A + \delta I$ and initial value $\Sigma(0) = x - x'$.

If the function $\psi(x) = \|x\|^\gamma$ would satisfy $A\psi \leq 0$, then $\psi(\Sigma(\tau \wedge t))e^{-\delta(\tau\wedge t)}$ would be a supermartingale which would readily lead to (32). As $\psi$ is not smooth, we consider the function $\psi_\beta(x) = \sum_{i=0}^n (x_i^2 + \beta)^{\gamma/2}$ with $\beta > 0$. We have :

$$\begin{aligned} A\psi_\beta &= \gamma(x_0^2+\beta)^{(\frac{\gamma}{2}-1)}((r-\frac{\delta}{\gamma})x_0^2 - \frac{\delta}{\gamma}\beta) \\ &+ \sum_{i=1}^n \gamma(x_i^2+\beta)^{(\frac{\gamma}{2}-2)}(x_i^4(\frac{1}{2}\sigma_i^2(\gamma-1) + \alpha_i - \frac{\delta}{\gamma}) + x_i^2\beta(\frac{1}{2}\sigma_i^2 + \alpha_i - \frac{2\delta}{\gamma}) - \frac{\delta}{\gamma}\beta^2). \end{aligned}$$

Assumption (A.1) implies

$$r - \frac{\delta}{\gamma} < 0 \quad \text{and} \quad \frac{1}{2}\sigma_i^2(\gamma-1) + \alpha_i - \frac{\delta}{\gamma} < 0.$$

Consequently, there exists a positive constant $C$ such that

$$A\psi_\beta \leq C\beta^{\gamma/2}.$$

Applying now Ito's formula to $\psi_\beta$, we obtain

$$E(e^{-\delta(\tau'\wedge\tau)}\psi_\beta(\Sigma(\tau'\wedge\tau))) \leq \psi_\beta(x-x') + \frac{C}{\delta}\beta^{\gamma/2}. \tag{34}$$

Taking the limit of (34) when $\beta$ goes to zero, and using

$$\psi(x) \leq \psi_0(x) \leq (n+1)\psi(x)$$

we get

$$E(e^{-\delta(\tau'\wedge\tau)}\psi(\Sigma(\tau'\wedge\tau))) \leq (n+1)\psi(x-x')$$

which leads together with (33) to the desired estimate (32). ∎

**Proposition 3.2** *The value function $V$ is concave in $\mathcal{S}$.*

**Proof** : The dynamic (1) is linear and the solvency region $\mathcal{S}$ is convex. Hence, for any $\theta$ in $[0,1]$, $x$ and $x'$ in $\mathcal{S}$, $\mathcal{P}$ in $\mathcal{U}(x)$ and $\mathcal{P}'$ in $\mathcal{U}(x')$, we have $\theta\mathcal{P}+(1-\theta)\mathcal{P}' \in \mathcal{U}(y)$ for $y = \theta x + (1-\theta)x'$ and

$$V(y) \geq J_y(\theta\mathcal{P}+(1-\theta)\mathcal{P}') = E\left\{\int_0^{+\infty} e^{-\delta t}u(\theta c(t)+(1-\theta)c'(t))dt\right\}.$$

Moreover, since $u$ is concave and positive we infer

$$V(y) \geq \theta J_x(\mathcal{P}) + (1-\theta)J_{x'}(\mathcal{P}').$$

Taking now the supremum over all $\mathcal{P}$ and $\mathcal{P}'$, we obtain that $V$ is concave. As a consequence, $V$ is Lipschitz continuous in $\mathring{\mathcal{S}}$. ∎

**Proposition 3.3** *$V$ is non decreasing with respect to $x_i$, for $i = 0,\ldots,n$.*

**Proof** : Let us denote explicitly by $s(t,x)$ the process $s(t)$ defined in (1) with initial value $x$ and by $\tau_x$ the exit time of $\mathring{\mathcal{S}}$ of $s(t,x)$. As

$$V(x) = \sup_{\mathcal{P}\in\mathcal{U}} E\int_0^{\tau_x} e^{-\delta t}u(c(t))dt,$$

and $u$ is positive, it is enough to prove the nondecreasing property of the stopping time $\tau_x$ for any control process $\mathcal{P}$.

Define $y(t,x)$ by

$$\begin{cases} y_0(t,x) &= e^{-rt}s_0(t,x) \\ y_i(t,x) &= e^{-(\alpha_i-\frac{1}{2}\sigma_i^2)t-\sigma_iW_i(t)}s_i(t,x) \quad i=1,\ldots,n. \end{cases}$$

The process $y(t,x)$ evolves according to

$$\begin{cases} dy_0(t,x) &= e^{-rt}\left(-c(t)dt+\sum_{i=1}^n(-(1+\lambda_i)d\mathcal{L}_i(t)+(1-\mu_i)d\mathcal{M}_i(t))\right) \\ dy_i(t,x) &= e^{-(\alpha_i-\frac{1}{2}\sigma_i^2)t-\sigma_iW_i(t)}(d\mathcal{L}_i(t)-d\mathcal{M}_i(t)) \end{cases} \tag{35}$$

and satisfies $y(0,x)=x$. Hence, we can write

$$y(t,x) = x + Y(t,\mathcal{P})$$

where $Y(t,\mathcal{P})$ depends only on $\mathcal{P}$. Consequently,

$$s(t,x) = (e^{rt}x_0, (e^{(\alpha_i-\frac{1}{2}\sigma_i^2)t+\sigma_i W_i(t)}x_i)_{i=1\cdots n}) + S(t,\mathcal{P}) \tag{36}$$

where $S(t,\mathcal{P})$ is a process which is independent of $x$.

Consider now $\tilde{x} \geq x$ (i.e. $\tilde{x}_i \geq x_i$, $\forall i = 0,\ldots,n$) and fix $\mathcal{P}$ in $\mathcal{U}$. We have from (36),

$$s(t,x) \leq s(t,\tilde{x})$$

and

$$\mathcal{W}(s(t,x)) \leq \mathcal{W}(s(t,\tilde{x}))$$

where $\mathcal{W}$ is defined in (3).

We have

$$\tau_{\tilde{x}} = \inf\{t \geq 0,\ \mathcal{W}(s(t,\tilde{x})) \leq 0\}.$$

For any $t > \tau_{\tilde{x}}$, there exists $t'$ such that $\tau_{\tilde{x}} < t' < t$ and $\mathcal{W}(s(t',\tilde{x})) \leq 0$, which implies $\mathcal{W}(s(t',x)) \leq 0$ and $t > t' \geq \tau_x$. Consequently, $\tau_{\tilde{x}} \geq \tau_x$ which leads to $V(\tilde{x}) \geq V(x)$. ■

### 3.3 Existence and Uniqueness results

First, we show that the value function $V$ is a viscosity solution of the variational inequality (11). The problem is reduced to prove a weak dynamic programming principle (see [12]).

**Lemma 3.2** *(i) The value function $V$ satisfies*

$$0 \leq V(x) \leq \varphi(x), \qquad \forall x \in \mathcal{S}, \tag{37}$$

*where $\varphi(x)$ is the super-solution defined in Lemma 3.1.*

*(ii) There exists $C > 0$ such that*

$$|J_x(\mathcal{P}) - J_{x'}(\mathcal{P})| \leq C\|x - x'\|^\gamma, \quad \forall x, x' \in \mathcal{S}, \quad \forall \mathcal{P} \in \mathcal{U}, \tag{38}$$

*where $J_x(\mathcal{P})$ is given in (9).*

**Proof** : Estimates (37) and (38) are readily obtained from the proof of Proposition 3.1. ■

**Proposition 3.4** *The weak dynamic programming principle is satisfied for the value function $V$, that is*

$$V(x) = \sup_{\mathcal{P}\in\mathcal{U}} E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}V(s((\theta\wedge\tau)^-))\right), \ \forall x \in \mathcal{S}, \tag{39}$$

*for any stopping time $\theta$.*

**Proof** : By means of the Markov property we have for all $\mathcal{P}$ in $\mathcal{U}$

$$E^{\mathcal{F}^{\theta\wedge\tau}}\int_0^{\tau} e^{-\delta t}u(c(t))dt = \int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}J_{s((\theta\wedge\tau)^-)}(\mathcal{P}'),$$

with $\mathcal{P}'$ equal to $\mathcal{P}$ "shifted" by $\theta\wedge\tau$. Note that $\mathcal{P}'$ may not be admissible. The correct method would be to proceed with admissible systems composed with a filtration $(\Omega, \mathcal{F}_t, P)$, a Wiener process $W = (W_i)_{i=1,\dots,n}$ in $\mathbb{R}^n$, and an admissible control process $\mathcal{P}$ and consider $V$ as the supremum of $J_x(\mathcal{P})$ over all admissible systems instead of the supremum over all admissible policies. We give here a formal proof. Rigorous proofs are given in [12, 25, 10, 18]. Thus,

$$\begin{aligned} J_x(\mathcal{P}) &= E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}J_{s((\theta\wedge\tau)^-)}(\mathcal{P}')\right) \\ &\leq E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}V(s((\theta\wedge\tau)^-))\right). \end{aligned}$$

By taking the supremum over all policies $\mathcal{P}$, we deduce one inequality side of (39). For the reverse inequality, we need to construct nearly optimal controls for each initial state $x$ in a measurable way. To that purpose, consider $\varepsilon > 0$ and $\{\mathcal{S}^k\}_{k=1}^{\infty}$ a sequence of disjoint subsets of $\mathcal{S}$ such that

$$\bigcup_{k=1}^{\infty} \mathcal{S}^k = \mathcal{S}, \qquad diam(\mathcal{S}^k) < \varepsilon.$$

For any $k$, take $x^k$ in $\mathcal{S}^k$ and $\mathcal{P}^k = (c^k, (\mathcal{L}_i^k, \mathcal{M}_i^k)_{i=1,\dots,n})$ in $\mathcal{U}$ such that

$$V(x^k) - \varepsilon \leq J_{x^k}(\mathcal{P}^k). \tag{40}$$

Now, for a given stopping time $\theta$ and an arbitrary policy $\mathcal{P}$ in $\mathcal{U}$, we define $\mathcal{P}^\theta = (c^\theta, (\mathcal{L}_i^\theta, \mathcal{M}_i^\theta)_{i=1,\dots,n})$ with

$$\begin{aligned} c^\theta(t) &= c(t)1_{t<\theta} + c^k(t-\theta)1_{t\geq\theta}, \\ \mathcal{L}_i^\theta(t) &= \mathcal{L}_i(t)1_{t<\theta} + (\mathcal{L}_i(\theta^-) + \mathcal{L}_i^k(t-\theta))1_{t\geq\theta}, \\ \mathcal{M}_i^\theta(t) &= \mathcal{M}_i(t)1_{t<\theta} + (\mathcal{M}_i(\theta^-) + \mathcal{M}_i^k(t-\theta))1_{t\geq\theta}, \end{aligned}$$

for $s(\theta) \in \mathcal{S}^k$. Using (38) and (40) we have

$$\begin{aligned} J_{s(\theta^-)}(\mathcal{P}^k) &= (J_{s(\theta^-)}(\mathcal{P}^k) - J_{x^k}(\mathcal{P}^k)) + J_{x^k}(\mathcal{P}^k) \\ &\geq -C\varepsilon^\gamma - \varepsilon + V(x^k) \\ &\geq -2C\varepsilon^\gamma - \varepsilon + V(s(\theta^-)). \end{aligned}$$

Denoting by I the right-hand side of (39), there exists a policy $\mathcal{P}$ such that

$$I - \varepsilon \leq E\left(\int_0^{\theta\wedge\tau} e^{-\delta t}u(c(t))dt + e^{-\delta(\theta\wedge\tau)}V(s((\theta\wedge\tau)^-))\right)$$

and in virtue of the Markov property we get

$$I - \varepsilon \leq J_x(\mathcal{P}^{\theta\wedge\tau}) + (2C\varepsilon^\gamma + \varepsilon)$$

and

$$I - 2C\varepsilon^\gamma - 2\varepsilon \leq J_x(\mathcal{P}^{\theta\wedge\tau}) \leq V(x),$$

which leads to (39). ■

**Corollary 3.1** *The value function $V(x)$ defined by (10) is the viscosity solution of the variational inequality (11-12).*

In the case of pure diffusion processes, this is a standard result in the theory of viscosity solutions (see [19]). For singular stochastic control problems, we refer to Fleming and Soner [12, chapter 8, theorem 5.1].

**Proposition 3.5** *Under assumptions (A.1) and (A.2), the value function $V$ is the unique viscosity solution of the variational inequality (11-12) in the class of continuous functions in $\mathcal{S}$ which satisfy*

$$|V(x)| \leq C(1 + \|x\|^\gamma) \qquad \forall x \in \mathcal{S}. \tag{41}$$

*More precisely, if $v$ is a viscosity subsolution and $v'$ is a viscosity supersolution of (11) which satisfy (41) and $v \leq v'$ on $\partial\mathcal{S}$, then $v \leq v'$ in $\mathcal{S}$.*

**Proof** : By Corollary 3.1 and equation (37), the value function $V$ is a viscosity solution of (11-12) and satisfies (41). To prove the uniqueness, we use the Ishii technique, in particular we adapt the proofs of Theorems 3.3 and 5.1 of [8]. They are themselves based on the following corollary of Theorem 3.2 of [8] :

**Theorem 3.2** *Let $V$ be an upper semicontinuous function and $V'$ a lower semicontinuous function in an open domain $\mathcal{O}$ of $\mathbb{R}^N$. Set $W(x,y) = V(x) - V'(x) - \frac{k}{2}|x-y|^2$ with $k > 0$. Suppose that $(\hat{x},\hat{y})$ is a local maximum of $W$. Then, there exist two matrices $X$ and $Y$ in $S^N$ such that*

$$(k(\hat{x}-\hat{y}), X) \in \bar{J}^{2,+}V(\hat{x}), \quad (k(\hat{x}-\hat{y}), Y) \in \bar{J}^{2,-}V'(\hat{y})$$

*and*

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3k \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{42}$$

■

In this statement, $|.|$ denotes the euclidian norm and $\bar{J}^{2,+}$ is defined as follows :

$$\begin{aligned}\bar{J}^{2,+}v(x) = \{(p,X) \in \mathbb{R}^N \times S^N, \ \exists (x_n, p_n, X_n) \in \mathcal{O} \times \mathbb{R}^N \times S^N, \\ (p_n, X_n) \in J^{2,+}v(x_n) \text{ and } (x_n, v(x_n), p_n, X_n) \underset{n\to\infty}{\longrightarrow} (x, v(x), p, X)\}.\end{aligned}$$

$\bar{J}^{2,-}$ is similarly defined. If $F$ is a continuous function in $S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathcal{O}$ satisfying (22), and $v$ is a viscosity subsolution of (21), we have

$$F(X, p, v(x), x) \geq 0 \ \ \forall (p,X) \in \bar{J}^{2,+}v(x), \ \ \forall x \in \mathcal{O}. \tag{43}$$

We consider now $v$ and $v'$ as in Proposition 3.5 and we argue by contradiction. Suppose that there exists $z$ in $\mathcal{S}$ such that $v(z) - v'(z) > 0$. Define the function $w_k$ in $\mathcal{S} \times \mathcal{S}$ as :

$$w_k(x,y) = v(x) - v'(y) - \frac{k}{2}|x-y|^2 - \varepsilon(\mathcal{W}_\nu(x)^{\gamma'} + \mathcal{W}_\nu(y)^{\gamma'})$$

with

$$\mathcal{W}_\nu(x) = x_0 + \sum_{i=1}^{n}(1-\nu_i)x_i$$

and denote

$$m_k = \sup_{(x,y)\in\mathcal{S}\times\mathcal{S}} w_k(x,y).$$

In the following, $C, C_1, C_2$ denote generic constants.

**Lemma 3.3** *For $\nu = (\nu_i)_{i=1,\dots,n}$ with $-\lambda_i < \nu_i < \mu_i$, there exist $C_1$ and $C_2 > 0$ such that*

$$C_1|x| \le \mathcal{W}_\nu(x) \le C_2|x| \ \ \forall x \in \mathcal{S}. \tag{44}$$

**Proof** : The second inequality of (44) is straightforward. To obtain the first inequality, we use the non negativity of $\mathcal{W}$ (defined in (3)) in $\mathcal{S}$ :

$$\mathcal{W}_\nu(x) = \mathcal{W}(x) - \sum_{i=1}^{n}\min((\nu_i-\mu_i)x_i, (\nu_i+\lambda_i)x_i) \ge C\sum_{i=1}^{n}|x_i| \ge 0.$$

Moreover,

$$|x_0| = |\mathcal{W}_\nu(x) - \sum_{i=1}^{n}(1-\nu_i)x_i| \le \mathcal{W}_\nu(x) + C\sum_{i=1}^{n}|x_i| \le C\mathcal{W}_\nu(x).$$

Consequently,

$$|x| \le C\mathcal{W}_\nu(x).$$

∎

We shall fix $\gamma' > \gamma$ such that the assumption (A.1) is still valid with $\gamma'$ instead of $\gamma$, and $\nu$ as in Lemma 3.3. This will ensure $m_k < \infty$ (see Lemma 3.4). On the other hand, we have :

$$m_k \ge \bar{m} = \sup_{x\in\mathcal{S}}\{v(x) - v'(x) - 2\varepsilon\mathcal{W}_\nu(x)^{\gamma'}\} \ge v(z) - v'(z) - 2\varepsilon\mathcal{W}_\nu(z)^{\gamma'}.$$

Hence, there exists $\varepsilon > 0$ such that $m_k \ge \bar{m} > 0$ for any $k$; in the following, we consider such $\varepsilon$.

**Lemma 3.4** *There exist $x_k, y_k$ in $\mathcal{S}$ such that*

$$m_k = w_k(x_k, y_k) < +\infty,$$

$$k|x_k - y_k|^2 \underset{k\to\infty}{\longrightarrow} 0, \tag{45}$$

*and*

$$m_k \underset{k\to\infty}{\longrightarrow} \bar{m} \equiv \sup_{x\in\mathcal{S}}\{v(x) - v'(x) - 2\varepsilon\mathcal{W}_\nu(x)^{\gamma'}\}. \tag{46}$$

**Proof** : Since $v$ and $v'$ satisfy (41), we have

$$m_k \leq C + \sup_{x \in \mathcal{S}} (C_1 |x|^\gamma - C_2 |x|^{\gamma'}) < +\infty$$

Let $(x^n, y^n)$ be a maximizing sequence :

$$w_k(x^n, y^n) \geq m_k - \frac{1}{n} \geq \bar{m} - \frac{1}{n}$$

which implies that

$$C_2 |x^n|^{\gamma'} - C_1 |x^n|^\gamma \leq C.$$

Hence, $x^n$ is bounded and similarly $y^n$ is bounded. Consequently, there exists a converging subsequence of $(x^n, y^n)$ and the limit $(x_k, y_k) \in \mathcal{S} \times \mathcal{S}$ realizes the maximum of $w_k$. As

$$v(x_k) - v'(y_k) - \varepsilon(\mathcal{W}_\nu(x_k)^{\gamma'} + \mathcal{W}_\nu(y_k)^{\gamma'}) = m_k + \frac{k}{2}|x_k - y_k|^2 \geq 0$$

for any $k$,we conclude that $x_k$, $y_k$ and $k|x_k - y_k|^2$ are bounded. Moreover, for any subsequence of $(x_k, y_k)$ converging to $(\hat{x}, \hat{y})$ when $k$ goes to infinity, we have $\hat{x} = \hat{y}$ and using $m_k \geq \bar{m}$, we get

$$\limsup_{k \to \infty} \frac{k}{2}|x_k - y_k|^2 \leq v(\hat{x}) - v'(\hat{x}) - 2\varepsilon \mathcal{W}_\nu(\hat{x})^{\gamma'} - \bar{m} \leq 0$$

Consequently (45) and (46) are satisfied. ■

Now, since $\bar{m} > 0$ and $v \leq v'$ on $\partial\mathcal{S}$, the limit $\hat{x}$ is in $\mathring{\mathcal{S}}$; then for any converging subsequence of $(x_k, y_k)$, we have $(x_k, y_k) \in \mathring{\mathcal{S}} \times \mathring{\mathcal{S}}$ for $k$ large. We apply now Theorem 3.2 with $V = v - \varepsilon \mathcal{W}_\nu{}^{\gamma'}$ and $V' = v' + \varepsilon \mathcal{W}_\nu{}^{\gamma'}$ at the point $(x_k, y_k)$ in $\mathring{\mathcal{S}} \times \mathring{\mathcal{S}}$ . We obtain that there exist $X, Y$ in $S^{n+1}$ satisfying (42) such that

$$(p_k, X_k) \equiv \left(k(x_k - y_k) + \varepsilon\gamma' \mathcal{W}_\nu(x_k)^{\gamma'-1}\hat{p}, X + \varepsilon\gamma'(\gamma' - 1)\mathcal{W}_\nu(x_k)^{\gamma'-2}A\right) \in \bar{J}^{2,+}v(x_k) \quad (47)$$

$$(p'_k, Y_k) \equiv \left(k(x_k - y_k) - \varepsilon\gamma' \mathcal{W}_\nu(y_k)^{\gamma'-1}\hat{p}, Y - \varepsilon\gamma'(\gamma' - 1)\mathcal{W}_\nu(y_k)^{\gamma'-2}A\right) \in \bar{J}^{2,-}v'(y_k) \quad (48)$$

with $\hat{p} = (1, 1 - \nu_1, \ldots, 1 - \nu_n)$ and $A = \hat{p}^t\hat{p}$.

Denote

$$F(X, p, r, x) = \max\left(F_0(X, p, r, x) + u^*(p_0), \max_{1 \leq i \leq n} G_i(p), \max_{1 \leq i \leq n} H_i(p)\right),$$

$$F_0(X, p, r, x) = \frac{1}{2}\sum_{i=1}^n \sigma_i^2 x_i^2 X_{ii} + \sum_{i=1}^n \alpha_i x_i p_i + r x_0 p_0 - \delta r,$$

$$G_i(p) = -(1 + \lambda_i)p_0 + p_i,$$

$$H_i(p) = (1 - \mu_i)p_0 - p_i,$$

where $X = (X_{ij})_{i,j=0,\ldots,n}$, $p = (p_i)_{i=0,\ldots,n}$.

Note that although $F$ is continuous, $F$ takes its values in $\mathbb{R} \cup \{+\infty\}$, since $F = +\infty$ when $p_0 \leq 0$. This leads to a difficulty to obtain a similar property as (3.14) in [8] and consequently straightforward application of the results of [8] can not be used. Moreover, as the discount factor $\delta$ appears only in the $F_0$ component of $F$ and not in $G_i$ and $F_i$, property (3.13) of [8], that is

$$F(X,p,r,x) - F(X,p,s,x) \leq -\lambda(r-s) \text{ for } r \geq s, \text{ with } \lambda > 0,$$

is not satisfied.

Using that $v$ is a viscosity subsolution and $v'$ is a viscosity supersolution of (11) (that is of $F(D^2v, Dv, v, x) = 0$ in $\mathring{\mathcal{S}}$), and using (47) and (48), we get :

$$F(X_k, p_k, v(x_k), x_k) \geq 0,$$
$$F(Y_k, p'_k, v'(y_k), y_k) \leq 0.$$

This last inequality implies $G_i(p'_k) \leq 0$ and $H_i(p'_k) \leq 0$ and by linearity of $G_i$ and $H_i$, we get

$$G_i(p_k) = G_i(p'_k) - \varepsilon\gamma'(\mathcal{W}_\nu(x_k)^{\gamma'-1} + \mathcal{W}_\nu(y_k)^{\gamma'-1})(\lambda_i + \nu_i) < 0$$

and

$$H_i(p_k) = H_i(p'_k) + \varepsilon\gamma'(\mathcal{W}_\nu(x_k)^{\gamma'-1} + \mathcal{W}_\nu(y_k)^{\gamma'-1})(\nu_i - \mu_i) < 0.$$

This leads to

$$F_0(X_k, p_k, v(x_k), x_k) + u^*((p_k)_0) \geq 0 \geq F_0(Y_k, p'_k, v'(y_k), y_k) + u^*((p'_k)_0).$$

Using now that $u^*$ is non increasing and $(p'_k)_0 < (p_k)_0$, we obtain

$$F_0(X_k, p_k, v(x_k), x_k) - F_0(Y_k, p'_k, v'(y_k), y_k) \geq 0.$$

Hence,

$$\begin{aligned} 0 \quad \leq \quad & \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2((x_k)_i^2 X_{ii} - (y_k)_i^2 Y_{ii}) + k(\sum_{i=1}^{n}\alpha_i((x_k)_i - (y_k)_i)^2 + r((x_k)_0 - (y_k)_0)^2) \\ & -\delta(v(x_k) - v'(y_k) - \varepsilon(\mathcal{W}_\nu(x_k)^{\gamma'} + \mathcal{W}_\nu(y_k)^{\gamma'})) \\ & +\varepsilon(f(x_k) + f(y_k)) \end{aligned} \tag{49}$$

with

$$\begin{aligned} f(x) \quad = \quad & \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 x_i^2 \gamma'(\gamma'-1)\mathcal{W}_\nu(x)^{\gamma'-2}A_{ii} \\ & +\sum_{i=1}^{n}\alpha_i x_i \gamma' \mathcal{W}_\nu(x)^{\gamma'-1}\hat{p}_i + r x_0 \gamma' \mathcal{W}_\nu(x)^{\gamma'-1}\hat{p}_0 \\ & -\delta\mathcal{W}_\nu(x)^{\gamma'}. \end{aligned}$$

Since $\gamma'$ is such that (A.1) is satisfied, $f(x) \leq 0 \;\; \forall x \in \mathbb{R}^{n+1}$. Using (42), we see that the first term of the r.h.s. of (49) is bounded by $Ck|x_k - y_k|^2$. Hence,

$$0 \leq Ck|x_k - y_k|^2 - \delta m_k \underset{k\to\infty}{\longrightarrow} -\delta\bar{m} < 0.$$

We thus get a contradiction and Proposition 3.5 is proven. ■

# 4 Change of variables

## 4.1 Reduction of the state dimension

The value function $V$ defined by (7) has the homothetic property [9]

$$\forall \rho > 0,\ V(\rho x) = \rho^\gamma V(x). \tag{50}$$

Consequently, the $(n+1)$-dimensional VI (11) satisfied by $V$ can be reduced to a $n$-dimensional VI by using the following homogeneous model, that is by considering the new state variables :

$$\begin{cases} \rho = x_0 + \sum_{i=1}^{n}(1-\mu_i)x_i & \text{(net wealth)} \\ y_i = \dfrac{(1-\mu_i)x_i}{\rho},\ i = 1,\dots,n & \text{(fraction of net wealth invested in stock } i\text{)} \end{cases} \tag{51}$$

and the new control variable

$$C = \frac{c}{\rho} \quad \text{(fraction of net wealth dedicated to consumption).} \tag{52}$$

The function $V(x)$ can be written as

$$\begin{aligned} V(x) &= V(\rho(1-\sum_{i=1}^{n} y_i), \frac{\rho y_1}{(1-\mu_1)}, \dots, \frac{\rho y_n}{(1-\mu_n)}) \\ &= \rho^\gamma W(y) \end{aligned} \tag{53}$$

where the function

$$W(y) = V(1-\sum_{i=1}^{n} y_i, \frac{y_1}{(1-\mu_1)}, \dots, \frac{y_n}{(1-\mu_n)}) \tag{54}$$

is defined in

$$\tilde{S} = \{y = (y_1,\dots,y_n) \in \mathbb{R}^n,\ 1 - \sum_{i=1}^{n} \frac{\lambda_i + \mu_i}{1-\mu_i}\{y_i\}^- \geq 0\}$$

with $\{y\}^- = \max(0, -y)$.

Using inequality (37) we deduce that the function $W$ is bounded :

$$0 \leq W(y) \leq \varphi(1-\sum_{i=1}^{n} y_i, \frac{y_1}{(1-\mu_1)}, \dots, \frac{y_n}{(1-\mu_n)}) \leq a \tag{55}$$

The function $W$ satisfies

$$\max(\tilde{A}W + u^*(BW), \max_{1\leq i\leq n} \tilde{L}_i W, \max_{1\leq i\leq n} \tilde{M}_i W) = 0 \tag{56}$$

in $\overset{\circ}{\tilde{\mathcal{S}}}$ with the boundary condition $W = 0$ on $\partial\tilde{\mathcal{S}}$, where

$$\begin{aligned}
\tilde{A}W &= \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 W}{\partial y_j \partial y_k} + \sum_{j=1}^{n} b_j \frac{\partial W}{\partial y_j} - \beta W, && (57)\\
BW &= \gamma W - \sum_{j=1}^{n} y_j \frac{\partial W}{\partial y_j}, && (58)\\
\tilde{L}_i W &= \frac{\partial W}{\partial y_i} - \left(\frac{\lambda_i + \mu_i}{1-\mu_i}\right) BW, && (59)\\
\tilde{M}_i W &= -\frac{\partial W}{\partial y_i} && (60)
\end{aligned}$$

and

$$\begin{aligned}
a_{jk} &= \frac{y_j y_k}{2}\sum_{i=1}^{n}\sigma_i^2(\delta_{ki} - y_i)(\delta_{ji} - y_i) && (61)\\
b_j &= y_j \sum_{i=1}^{n}[(\gamma-1)\sigma_i^2 y_i + \alpha_i - r](\delta_{ij} - y_i), && (62)\\
\beta &= \delta - \gamma(r + \sum_{i=1}^{n}[(\alpha_i - r)y_i + \frac{\gamma-1}{2}\sigma_i^2 y_i^2]). && (63)
\end{aligned}$$

The symbol $\delta_{ij}$ denotes the Kronecker index which is equal to 0 when $i \neq j$ and equal to 1 when $i = j$.

**Remark 4.1** Equation (56) only depends on $\nu = (\nu_i)_{i=1\dots n}$ with $\nu_i = \dfrac{\lambda_i + \mu_i}{1-\mu_i}$, and so does the function $W$. Let us denote by $V_{\lambda,\mu}$ the value function (7) in order to express explicitly the dependency of $V$ on the transaction costs and by $W_{\lambda,\mu}$ the solution of (56). We have :

$$\begin{aligned}
W_{\lambda,\mu}(y) &= W_{\nu,0}(y)\\
&= V_{\nu,0}(1 - \sum_{i=1}^{n} y_i, y_1, \dots, y_n).
\end{aligned} \qquad (64)$$

Using (50),we get

$$V_{\lambda,\mu}(x) = V_{\nu,0}(x_0, (1-\mu_1)x_1, \dots, (1-\mu_n)x_n). \qquad (65)$$

Consequently, it is sufficient to compute the value function $V$ when the transaction costs on sale are equal to zero. ■

Using the properties of $V$ and (56), we deduce that $W$ is concave, non negative and non decreasing with respect to each coordinate $y_i$.

## 4.2 Additional treatment for numerical purpose

Our purpose is now to solve equation (56).

In order to simplify the numerical computation, we restrict the admissible region $\mathcal{S}$ to

$$\mathcal{S}^+ = \{x \in \mathbb{R}^{n+1},\ x_1, \dots, x_n \geq 0,\ x_0 + \sum_{i=1}^{n} (1-\mu_i) x_i \geq 0\}$$

that is, we suppose that the amounts of money allocated in the risky assets are non negative, while the amount of money in the bank account can be negative as long as the net wealth remains non negative. This is not restrictive since, when $\alpha_i > r$, the no-transaction cone is inside $\mathcal{S}^+$ and a trajectory which starts in $\mathcal{S}^+$ remains in $\mathcal{S}^+$(see [9] for $n = 1$).

This leads to the study of the VI (56) in the domain $(\mathbb{R}^+)^n$. No boundary condition has to be specified since the VI degenerates on $\{y_i = 0\}$, $i = 1, \dots, n$. Note that the function $W$ has bounded derivatives in $(\mathbb{R}^+)^n$.

We proceed with a technical change of variables which brings $(\mathbb{R}^+)^n$ to $[0,1]^n$, namely :

$$\begin{cases} \psi(z) = \theta(z) W(y), \\ \theta(z) = \prod_{i=1}^{n} (1 - z_i), \\ z_i = \dfrac{y_i}{1 + y_i}, \quad i = 1, \dots, n. \end{cases} \tag{66}$$

The function $\psi$ is bounded and concave with respect to $z_i$, $i = 1, \dots, n$, has bounded derivatives, and satisfies :

$$\begin{cases} \max(\bar{A}\psi + \sup_{C \geq 0} (-C\bar{B}\psi + \theta(z)\dfrac{C^\gamma}{\gamma}), \max_{1 \leq i \leq n} \bar{L}_i\psi, \max_{1 \leq i \leq n} \bar{M}_i\psi) = 0 \quad \text{in } (0,1)^n \\ \psi = 0 \text{ on } [0,1]^n \cap \{z_i = 1\} \quad \forall i = 1, \dots, n \end{cases} \tag{67}$$

where

$$\begin{aligned} \bar{A}\psi &= \sum_{j,k=1}^{n} a'_{jk} \frac{\partial^2 \psi}{\partial z_j \partial z_k} + \sum_{j=1}^{n} b'_j \frac{\partial \psi}{\partial z_j} - \beta' \psi, \\ \bar{B}\psi &= (\gamma - \sum_{j=1}^{n} z_j)\psi - \sum_{j=1}^{n} z_j (1 - z_j) \frac{\partial \psi}{\partial z_j}, \\ \bar{L}_i\psi &= (1 - z_i)(\psi + (1 - z_i)\frac{\partial \psi}{\partial z_i}) - \lambda_i \bar{B}\psi, \\ \bar{M}_i\psi &= -(1 - z_i)(\psi + (1 - z_i)\frac{\partial \psi}{\partial z_i}), \end{aligned}$$

with

$$
\begin{aligned}
a'_{jk} &= \frac{1}{2}z_j(1-z_j)z_k(1-z_k)\bar{a}_{jk}, \\
b'_j &= z_j(1-z_j)(\bar{b}_j + \sum_{\substack{k=1 \\ k\neq j}}^{n} z_k\bar{a}_{jk}), \\
\beta' &= \beta - \sum_{j=1}^{n} z_j\bar{b}_j - \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \bar{a}_{jk}\frac{z_jz_k}{2}, \\
\bar{a}_{jk} &= \sum_{i=1}^{n} \sigma_i^2(\delta_{ki} - \frac{z_i}{1-z_i})(\delta_{ji} - \frac{z_i}{1-z_i}), \\
\bar{b}_j &= \sum_{i=1}^{n}((\gamma-1)\sigma_i^2\frac{z_i}{1-z_i} + \alpha_i - r)(\delta_{ij} - \frac{z_i}{1-z_i}),
\end{aligned}
$$

and $\beta$ defined in (63).

The numerical study is organized as follows : equation (67) is solved by using the numerical methods explained in section 5 below. Then a reverse change of variable is performed in order to display the numerical results for problem (56) (see section 6).

# 5 Numerical methods

We consider equations of the form :

$$
\begin{cases} \max\limits_{P \in \mathcal{P}_{ad}} (A^P W + u(P)) = 0 & \text{in } \Omega = (0,1)^m \\ W = 0 & \text{on } \Gamma \end{cases} \tag{68}
$$

where $A^P$ is a second order degenerated elliptic operator

$$
A^P W(x) = \sum_{i,j=1}^{m} a_{ij}(x,P)\frac{\partial^2 W}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{m} b_i(x,P)\frac{\partial W}{\partial x_i}(x) - \beta(x,P)W(x)
$$

with

$$
\sum_{i,j=1}^{m} a_{ij}(x,P)\eta_i\eta_j \geq 0, \qquad \beta(x,P) \geq 0 \qquad \forall x \in \Omega,\ \eta \in \mathbb{R}^m,\ P \in \mathcal{P}_{ad}.
$$

$\mathcal{P}_{ad}$ is a closed subset of $\mathbb{R}^k$ and $\Gamma$ is a part of the boundary $\partial\Omega$, which consists of faces of the $m$-cube $\Omega$. In $\partial\Omega \setminus \Gamma$, the operator $A^P$ is degenerated for any $P$ and no boundary condition is needed.

Bellman equations are clearly equations of this type, whereas variational inequalities like (67) can also be formulated in this form by using an additive discrete control which selects the equation which satisfies the maximum.

The numerical study of equation (68) consists in two steps : first, discretize (68) by using the finite difference method, and then solve the discrete equation by means of an iterative method. This procedure and the computer implementation are treated by using the expert system *Pandore* [5, 2] that has been developed to automate studies in stochastic control.

## 5.1 Discretization

Let $h = 1/N$ ($N \in \mathbb{N}^*$) denote the finite difference step in each coordinate direction, $e_i$ the unit vector in the $i^{th}$ coordinate direction, and $x = (x_1, \ldots, x_m)$ a point of the uniform grid $\Omega_h = \bar{\Omega} \cap (h\mathbb{Z})^m \setminus \Gamma$. Equation (68) is discretized by replacing the first and second order derivatives of $W$ by the following approximation :

$$\frac{\partial W}{\partial x_i} \sim \frac{W(x+he_i) - W(x-he_i)}{2h} \tag{69}$$

or

$$\frac{\partial W}{\partial x_i} \sim \begin{cases} \dfrac{W(x+he_i) - W(x)}{h} & \text{when} \quad b_i(x,P) \geq 0 \\ \dfrac{W(x) - W(x-he_i)}{h} & \text{when} \quad b_i(x,P) < 0. \end{cases} \tag{70}$$

$$\frac{\partial^2 W}{\partial x_i^2}(x) \sim \frac{W(x+he_i) - 2W(x) + W(x-he_i)}{h^2}, \tag{71}$$

$$\begin{aligned} \frac{\partial^2 W}{\partial x_i \partial x_j}(x) &\sim \frac{W(x+he_i+he_j) - W(x+he_i-he_j)}{4h^2} \\ &+ \frac{W(x-he_i-he_j) - W(x-he_i+he_j)}{4h^2} \qquad \text{for } i \neq j. \end{aligned} \tag{72}$$

Approximation (69) may be used when $A$ is uniformly elliptic, whereas (70) has to be used when $A$ is degenerated (see [17]). These differences are computed in the entire grid $\Omega_h$, by prolonging $W$ on the "boundary" of $\Omega_h$ in $(h\mathbb{Z})^m$ :

$$\begin{aligned} W(x) &= 0 & \forall x \in \Gamma \cap (h\mathbb{Z})^m \\ W(x-he_i) &= W(x) & \forall x \in \{x_i = 0\} \cap \Omega_h \\ W(x+he_i) &= W(x) & \forall x \in \{x_i = 1\} \cap \Omega_h. \end{aligned}$$

We obtain a system of $N_h$ non linear equations of $N_h$ unknowns $\{W_h(x),\ x \in \Omega_h\}$ :

$$\max_{P \in \mathcal{P}_{ad}} (A_h^P W_h + u(P))(x) = 0, \quad \forall x \in \Omega_h \tag{73}$$

where $N_h = \sharp\Omega_h \sim 1/h^m$. Let $\mathcal{P}_h$ denote the set of control functions $P : \Omega_h \longrightarrow \mathcal{P}_{ad}$ and $\mathcal{V}_h$ the set of functions from $\Omega_h$ into $\mathbb{R}$. Equation (73) can be rewritten :

$$\max_{P \in \mathcal{P}_h} (A_h^P W_h + u(P)) = 0, \quad W_h \in \mathcal{V}_h.$$

Then, the operator $A_h^P$, depending on $P$ in $\mathcal{P}_h$, maps $\mathcal{V}_h$ into itself (or is a $N_h \times N_h$ matrix).

Because of the degeneracy of the operator $A^P$ at some points of the closed $m$-cube $\bar{\Omega}$ and the presence of mixed derivatives, $A_h^P$ does not satisfy the usual Discrete Maximum Principle (i.e. $(A_h^P W_h(x) \leq 0,\ \ \forall x \in \Omega_h) \Rightarrow (W_h(x) \geq 0,\ \ \forall x \in \Omega_h)$). Consequently, equation (73) may not be stable, even for small step $h$. However $A_h^P$ can be written as the sum of a symmetric negative definite operator and an operator which satisfies the Discrete Maximum Principle; we thus infer the stability of $A_h^P$ which is confirmed by numerical experiments.

Two algorithms are available to solve equation (73), *(i)* the classical value iteration (successive approximation) algorithm and *(ii)* the (Full) Multigrid-Howard algorithm based on the "Howard algorithm" (policy iteration) and the multigrid method [1, 2]. The former is simpler but has a larger complexity.

Both algorithms are described below but the numerical tests given in the following sections are performed by using the second one only.

## 5.2 The Value Iteration method

Suppose that the $N_h \times N_h$ matrix $A_h^P$ satisfies

$$(A_h^P)_{ij} \geq 0, \quad \forall i \neq j, \quad \sum_{j=1}^{N_h} (A_h^P)_{ij} = -\lambda < 0, \quad \forall i \tag{74}$$

which implies that $A_h^P$ satisfies the Discrete Maximum Principle. Equation (73) can be rewritten as

$$W_h = \frac{1}{1+\lambda k} \max_{P \in \mathcal{P}_h} (M^P W_h + k u(P)), \tag{75}$$

where $k > 0$ and $M^P = I + k(A_h^P + \lambda I)$ is a Markov matrix ($I$ is the $N_h \times N_h$ identity matrix). Equation (73) can then be interpreted as the Dynamic Programming equation of a control problem of Markov chain with discount factor $1/(1+\lambda k)$, instantaneous cost $ku(P)$ and transition matrix $M^P$

$$\max_{(P_n)} \sum_{n=0}^{\tau} \frac{k}{(1+\lambda k)^{n+1}} u(X_n, P_n).$$

The value iteration method [4] consists in the contraction iteration

$$W^{n+1} = \frac{1}{1+\lambda k} \max_{P \in \mathcal{P}_h} (M^P W^n + k u(P)). \tag{76}$$

The contracting factor is $1/(1+\lambda k) = 1 - \mathcal{O}(h^2)$ and the complexity[1] of the method is

$$C_h = \mathcal{O}(\frac{-\log h}{h^2} N_h) = \mathcal{O}(-h^{-(2+m)} \log h) = \mathcal{O}(N_h^{1+2/m} \log N_h).$$

When the operator $A_h^P$ does not satisfy the Discrete Maximum Principle, equation (73) cannot be interpreted as a discrete Bellman equation. Nevertheless, the iterative method (76) can still be used if we find $\lambda$ and $k$ such that the $L^2$ norm of $M^P$ (which is no more a Markov matrix) is lower than 1 for all $P$. This condition may be obtained for instance when the discount factor $\beta(x, P)$ is large enough.

[1] The number of elementary operations for computing an approximation of the solution of (73) with an error in the order of the discretization error.

## 5.3 The Multigrid-Howard algorithm

Another classical algorithm is the Howard algorithm [16, 3, 4] also named policy iteration. It consists in an iteration algorithm on the control and value functions (starting from $P^0$ or $W^0$) :

$$\text{for } n \geq 1 \qquad P^n \in \underset{P \in \mathcal{P}_h}{\operatorname{Argmax}}(A_h^P W^{n-1} + u(P)) \tag{77}$$

$$\text{for } n \geq 0 \qquad W^n \text{ is solution of } A_h^{P^n} W + u(P^n) = 0. \tag{78}$$

When $A_h^P$ satisfies the Discrete Maximum Principle, the sequence $W^n$ decreases and converges to the solution of (73) and the convergence is in general superlinear (see [3, 4, 1, 2]).

The exact computation of step (78) is expensive in dimension $m \geq 2$ (the complexity of a direct method is $\mathcal{O}(N_h^{3-2/m})$). We thus use the multigrid-Howard algorithm introduced in [1, 2] : in (78), $W^n$ is computed by a multigrid method with initial value $W^{n-1}$. The advantage is that each multigrid iteration takes a computing time of $\mathcal{O}(N_h)$ and contracts the error by a factor independent of the discretization step $h$. For a detailed description of the multigrid algorithm, see for example [21, 15, 14].

Let $\mathcal{M}^P$ denote the operator of an iteration of the multigrid method associated to the equation $A_h^P W + u(P) = 0$. Starting from $W^0$, we proceed the following iteration :

$$\left\{ \begin{array}{l} (77) \\ \text{for } n \geq 1 \left\{ \begin{array}{l} W^{n,0} = W^{n-1} \\ \text{for } i = 1 \text{ to } m_n, \;\; W^{n,i} = \mathcal{M}^{P^n}(W^{n,i-1}) \\ W^n = W^{n,m_n} \end{array} \right. \end{array} \right. \tag{79}$$

This algorithm is converging to the solution $W_h^*$ of (73) if $W^0$ is sufficiently close to $W_h^*$ and $m_n$ is large enough (independently of the step $h$) [1, 2].

Let us now introduce the FMGH algorithm which solves equation (73) from any initial value $W^0$.

## 5.4 The Full-Multigrid-Howard algorithm

This algorithm fully uses the idea of the Full-Multigrid method (FMG) (see [1, 2]).

Let us consider the sequence of grids $(\Omega_k)_{k\geq 1}$ of steps $h_k = 2^{-k}$ and denote by $\mathcal{I}_k^{k+1}$ the operator of the $m$-linear interpolation from $\mathcal{V}_{h_k}$ into $\mathcal{V}_{h_{k+1}}$.

If $W_k \in \mathcal{V}_{h_k}$, $W_{k+1} = \mathcal{I}_k^{k+1} W_k$ is defined by :

$$\begin{array}{rcll} W_{k+1}(x) & = & W_k(x) & \forall x \in \Omega_k \subset \Omega_{k+1} \\ W_{k+1}(\frac{x+y}{2}) & = & \dfrac{W_{k+1}(x) + W_{k+1}(y)}{2} & \forall x, y \in \Omega_{k+1} \text{ such that } \frac{x+y}{2} \in \Omega_{k+1} \\ & & & \text{and } x \text{ and } y \text{ are in the same cell of } \Omega_k, \end{array}$$

where a cell of $\Omega_h$ is a $m$-cube of width $h$ included in $\bar{\Omega}$ and with vertices in $(h\mathbb{Z})^m$.

The Full-Multigrid-Howard (FMGH) algorithm is defined as :

For $1 \leq k \leq \bar{k}$,

$W_k^{\bar{n}}$ is the $\bar{n}$-th iteration of the sequence defined by (79) in the grid $\Omega_k$ of initial value $W_k^0$.

For $1 \leq k < \bar{k}$,

$$W_{k+1}^0 = \mathcal{I}_k^{k+1} W_k^{\bar{n}}.$$

Under appropriate assumptions (see [1, 2]), the error between $W_k^{\bar{n}}$ and the solution $W_k^*$ of (73) with $h = h_k$ is in the order of the discretization error, for any $k$. This property is realized for any initial value $W_1^0$, if the numbers $m_n$ and $\bar{n}$ are large enough (but independent of the level $k$). Consequently, this algorithm solves equation (73) (with an error in the order of the discretization error) with a computing time of $\mathcal{O}(N_h)$.

# 6 Numerical results

Equation (56) is solved in $(\mathbb{R}^+)^n$ for $n = 1$ and $n = 2$ and various numerical values of the parameters. Numerical results are gathered at the end of the paper. We recall that, because of (65), we can consider, without restriction, the case $\mu = 0$.

We are mainly interested in the determination of the regions where the different transactions take place.

**Remark 6.1** The regions $B_i$ and $S_i$ defined in (17) and (18) are characterized by

$$\begin{aligned} B_i &= \{x \in \mathcal{S},\ \tilde{L}_i W(y) = 0,\ y \text{ given by (51)}\}, \\ S_i &= \{x \in \mathcal{S},\ \tilde{M}_i W(y) = 0,\ y \text{ given by (51)}\}. \end{aligned}$$

where the operators $\tilde{L}_i$ and $\tilde{M}_i$ are defined in (59) and (60).

By extension we use the notation :

$$\begin{aligned} & B_i = \{y \in (\mathbb{R}^+)^n,\ \tilde{L}_i W(y) = 0\}, \\ & S_i = \{y \in (\mathbb{R}^+)^n,\ \tilde{M}_i W(y) = 0\}, \\ & NT_i = (\mathbb{R}^+)^n \backslash (B_i \cup S_i) \\ & NT = \bigcap_{i=1}^n NT_i. \end{aligned} \tag{80}$$

■

## 6.1 One risky asset

Numerical tests are performed with $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\lambda = \lambda_1 = 2, 4, 6, 8$ or $10\%$.

The value function $W$, solution of (56), and the optimal consumption $C$ are displayed in Figures 3 and 4.

The regions $B_1$ and $S_1$ are of the form (see section 7) : $B_1 = [0, \pi^-]$ and $S_1 = [\pi^+, +\infty)$ with $0 < \pi^- < \pi^+$. The function $1_{B_1} - 1_{S_1}$ is displayed in Figure 5. It is equal to 1 in $B_1$, 0 in $NT_1$ and $-1$ in $S_1$.

The functions $\pi^+$ and $\pi^-$ are displayed in Figure 6 as functions of $\lambda$. The initial point $\pi^+(0) = \pi^-(0)$ has been set to $\pi_1^*$, where $\pi_1^*$ is the optimal proportion of risky asset in the Merton problem (defined below).

These results are similar to those obtained by Davis and Norman [9].

## 6.2 Two risky assets

We set $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$ and fix the parameters of the first risky asset to $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\lambda_1 = 2\%$.

Four tests are performed with

$$\begin{array}{llll} \text{test 1 :} & \alpha_2 = 15\% & \sigma_2 = 35\% & \lambda_2 = 5\%, \\ \text{test 2 :} & \alpha_2 = 15\% & \sigma_2 = 35\% & \lambda_2 = 2\%, \\ \text{test 3 :} & \alpha_2 = 15\% & \sigma_2 = 35\% & \lambda_2 = 10\%, \\ \text{test 4 :} & \alpha_2 = 20\% & \sigma_2 = 50\% & \lambda_2 = 5\%. \end{array}$$

For test 1, the value function $W$, the optimal consumption $C$ and their contour lines are displayed in Figures 7, 8, 9 and 10. Then the partition of the domain is displayed for each test in Figures 11, 12, 13 and 14.

Note that, at first sight, the boundaries of the regions $B_i$ and $S_i$ seem to be straight lines of equation $y_i =$ constant. This would mean that the investment policies are decoupled, although the dynamics are correlated. In fact, when the cost for purchase $\lambda_2$ grows, the region $NT_2$ grows as expected but the boundaries of $S_1$ and $B_1$ are also perturbed. Moreover, a variation of $\alpha_2$ and $\sigma_2$ affect both $NT_2$ and $NT_1$. A theoretical study of the boundaries is done below, in order to confirm these remarks.

# 7 Theoretical analysis of the optimal strategy

## 7.1 No transaction costs : the Merton problem

When the transaction costs are equal to zero, the optimal investment strategy is to keep a constant fraction of total wealth in each risky asset (see [23, 9, 12, 26]). Indeed, set $\lambda = \mu = 0$ in equation (56). We obtain :

$$\max(\tilde{A}W + u^*(BW), \max_{1 \le i \le n} \frac{\partial W}{\partial y_i}, \max_{1 \le i \le n}(-\frac{\partial W}{\partial y_i})) = 0 \tag{81}$$

which is equivalent to

$$\begin{cases} W = \text{ constant}, \\ -\beta(y)W + u^*(\gamma W) \leq 0, \quad \forall y \in (\mathbb{R}^+)^n, \end{cases} \tag{82}$$

with $\beta(y)$ defined in (63). Uniqueness of the solution of (82) is not insured since assumption (A.2) is not satisfied, but the function $W$ defined in (54) is the minimal solution of VI (81). Hence, we have

$$\max_{y \in (\mathbb{R}^+)^n} \{-\beta(y)W + u^*(\gamma W)\} = 0. \tag{83}$$

Equation (83) coincides with the Bellman equation of the problem where the proportion $y_i$ is considered as a control variable [9]. Under assumption (A.1), the optimal proportion, denoted by $\pi_i^*$ and called the "Merton proportion" is given by

$$\pi_i^* = \frac{\alpha_i - r}{\sigma_i^2(1-\gamma)}.$$

The optimal fraction of wealth dedicated to consumption is

$$C^* = \frac{1}{1-\gamma}\left(\delta - \gamma\left(r + \frac{1}{2(1-\gamma)}\sum_{i=1}^{n}\left(\frac{\alpha_i - r}{\sigma_i}\right)^2\right)\right),$$

and the value function $W$ is equal to

$$W = \frac{C^{*(\gamma-1)}}{\gamma}.$$

The regions "sell $i$" and "buy $i$" are characterized by

$$\begin{aligned} B_i &= \{y \in (\mathbb{R}^+)^n, \ y_i \leq \pi_i^*\}, \\ S_i &= \{y \in (\mathbb{R}^+)^n, \ y_i \geq \pi_i^*\}. \end{aligned}$$

Note that these regions are not obtained by merely setting $\lambda = \mu = 0$ in equation (80) but by taking the limit of these expressions when $\lambda$ and $\mu$ tend to 0.

## 7.2 A general shape of the transaction regions

In this section, we derive formally from VI (56), without numerical computation, the general shape of the transaction regions, given in Figure 1. To that purpose, we assume the function $W$ to be $\mathcal{C}^2$ in the interior of $(\mathbb{R}^+)^n$. Although this is not true in general, what is done below can be adapted by using the theory of viscosity solutions. This approach is used for example in [12] to obtain regularity results for the value function $V$ and general properties of the transaction regions for $n = 1$.

From (56), we have $\tilde{M}_i W \leq 0$; besides, the concavity of $W$ implies that $\tilde{M}_i W = -\dfrac{\partial W}{\partial y_i}$ is non decreasing with respect to $y_i$. Consequently, the region $S_i$, defined in (80) can be written as

$$S_i = \{y \in (\mathbb{R}^+)^n, \ y_i \geq \pi_i^+(\hat{y})\}$$

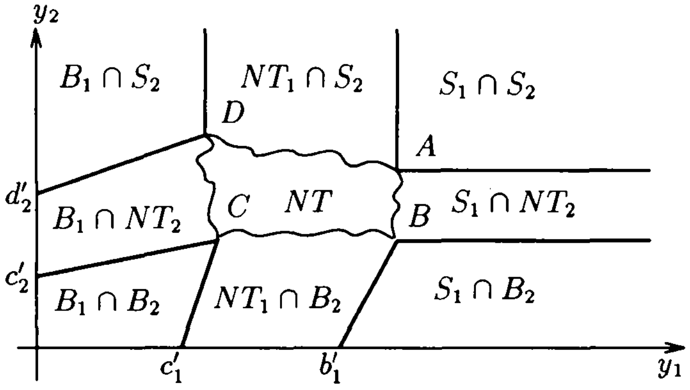


Figure 1: General shape of the transaction regions

where $\pi_i^+$ is some mapping of

$$\hat{y} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n).$$

To obtain a similar characterization for $B_i$, we consider another change of variables $(\rho', y')$ obtained by substituting $-\lambda_i$ for $\mu_i$ in (51) for some fixed $i \in \{1, \dots, n\}$. Proceeding as above, and using Remark 6.1, we obtain :

$$B_i = \{y \in (\mathbb{R}^+)^n,\ y'_i \le \pi_i'^-(\hat{y}')\} \tag{84}$$

with

$$y'_i = \frac{(1+\lambda_i)y_i}{1+\lambda_i y_i} \quad \text{and} \quad \hat{y}' = \frac{1}{1+\lambda_i y_i}\hat{y}. \tag{85}$$

Since $y'_i$ is non decreasing with respect to $y_i$, we get

$$B_i = \{y \in (\mathbb{R}^+)^n,\ y_i \le \pi_i^-(\frac{1}{1+\lambda_i y_i}\hat{y})\}.$$

We suppose $\pi_i^+ < +\infty$ and $\pi_i^- > 0$ which imply that $\pi_i^+$ and $\pi_i^-$ are continuous functions and that the regions $S_i$ and $B_i$ are connected.

We restrict now to the case $n = 2$, but what is done below can easily be generalized to $n > 2$.

In $S_1$, $\tilde{M}_1 W = -\dfrac{\partial W}{\partial y_1} = 0$. The function $W$ is thus constant with respect to $y_1$ in $S_1$. Consequently the pieces of boundaries $\partial B_2$ and $\partial S_2$ included in $S_1$ are straight lines of equation $y_2 =$ constant. Similarly, using the change of variables (85) with $i = 1$, we infer that the pieces of boundaries $\partial B_2$ and $\partial S_2$ included in $B_1$ are straight lines of equation

$$y'_2 = \frac{y_2}{1+\lambda_1 y_1} = \text{constant}.$$

By symmetry, we get similar properties for the boundaries $\partial B_1$ and $\partial S_1$ as displayed in Figure 1. No additive property has been obtained for the boundary of $NT$.

A question which arises now is how is located the "Merton proportion" $\pi^*$. In general, $\pi^*$ is not necessarily in the region $NT$. Nevertheless, we have

**Proposition 7.1** *We use the notation of Figure 1 :*

$$A = (a_1, a_2) = \partial S_1 \cap \partial S_2, \qquad B = (b_1, b_2) = \partial S_1 \cap \partial B_2,$$

$$C = (c_1, c_2) = \partial B_1 \cap \partial B_2, \qquad D = (d_1, d_2) = \partial B_1 \cap \partial S_2,$$

$$c_1' = \frac{c_1}{1+\lambda_2 c_2}, \quad b_1' = \frac{b_1}{1+\lambda_2 b_2}, \quad c_2' = \frac{c_2}{1+\lambda_1 c_1}, \quad d_2' = \frac{d_2}{1+\lambda_1 d_1}.$$

*and set*

$$\tilde{\pi}_i^* = \begin{cases} \dfrac{\pi_i^*}{1+\lambda_i - \lambda_i \pi_i^*} & \text{if } \ \pi_i^* < 1 + \dfrac{1}{\lambda_i}, \\ +\infty & \text{otherwise.} \end{cases}$$

*Then, we have*

$$\pi_1^* \le a_1, b_1', \quad \pi_2^* \le a_2, d_2'$$

*and*

$$d_1, c_1' \le \tilde{\pi}_1^*, \quad b_2, c_2' \le \tilde{\pi}_2^*.$$

**Proof** : We prove $\pi_i^* \le a_i$, $i = 1, 2$. The other inequalities are obtained similarly by using the change of variables (85). In $S_1 \cap S_2$, the function $W$ is equal to a constant $W_0$ and satisfies (56) which reduces to :

$$-\beta(y) W_0 + u^*(\gamma W_0) \le 0$$

with $\beta(y)$ given in (63).

Hence,

$$-\beta(y) + (1-\gamma)\gamma^{\frac{1}{\gamma-1}} W_0^{\frac{1}{\gamma-1}} \le 0 \quad \forall y \in S_1 \cap S_2.$$

On the other hand, the point $A$ is in $S_1 \cap S_2 \cap \overline{NT}$. Assuming that $W$ is $C^2$ at point $A$, we obtain

$$\tilde{A}W + u^*(BW) = 0 \tag{86}$$

and

$$-\beta(A) + (1-\gamma)\gamma^{1/\gamma-1} W_0^{\frac{1}{\gamma-1}} = 0.$$

Consequently

$$\beta(y) \ge \beta(A) \quad \forall y \in S_1 \cap S_2 = [a_1, +\infty) \times [a_2, +\infty).$$

As the function $\beta(y)$ is of the form $\beta_1(y_1) + \beta_2(y_2)$ with quadratic functions $\beta_i$, we get

$$\beta_i(y_i) \ge \beta_i(a_i) \quad \forall y_i \ge a_i.$$

Consequently, $a_i \ge \operatorname{Argmin} \beta_i = \pi_i^*$. ∎

### 7.3 Special case of no transaction cost for one of the risky assets

We suppose here $n = 2$, $\lambda_1 = \mu_1 = 0$, $\lambda_2 \neq 0$, $\mu_2 = 0$. The VI (56) then reduces to

$$\max(\tilde{A}W + u^*(BW), \frac{\partial W}{\partial y_1}, \frac{\partial W}{\partial y_2} - \lambda_2 BW, -\frac{\partial W}{\partial y_1}, -\frac{\partial W}{\partial y_2}) = 0 \tag{87}$$

which implies that the function $W$ is independent of $y_1$. Consequently the boundaries of $B_2$ and $S_2$ are horizontal straight lines of equation $y_2 = \pi_2^-$, $y_2 = \pi_2^+$ respectively. Since equation (87) holds for all $y_1$, and $W$ is the minimal solution, we have

$$\max(\max_{y_1 \geq 0}(\tilde{A}W + u^*(BW)), \frac{\partial W}{\partial y_2} - \lambda_2 BW, -\frac{\partial W}{\partial y_2}) = 0.$$

The regions $B_1$ and $S_1$ are delimited by the curve of equation $y_1 = \pi_1(y_2)$ where $\pi_1(y_2) = \operatorname*{Argmax}_{y_1 \geq 0}(\tilde{A}W + u^*(BW))$ is the solution of

$$y_1\sigma_1^2 y_2^2 \frac{\partial^2 W}{\partial y_2^2} - (2(\gamma-1)\sigma_1^2 y_1 + (\alpha_1 - r))y_2 \frac{\partial W}{\partial y_2} + \gamma((\alpha_1 - r) + (\gamma - 1)\sigma_1^2 y_1)W = 0.$$

Consequently

$$\pi_1(y_2) = \frac{\pi_1^* BW}{BW + \dfrac{y_2}{1-\gamma}\dfrac{\partial BW}{\partial y_2}}.$$

In particular $\pi_1(0) = \pi_1^*$ and $\pi_1(y_2) = (1 + \lambda_2 y_2)\pi_1^*$ in $B_2$. In $S_2$, $W$ is constant and $\pi_1(y_2) = \pi_1^*$ (see Figure 2). Moreover, by using the concavity of $W$, we obtain the estimate

$$0 < \pi_1(y_2) \leq \frac{\pi_1^*}{1 - \lambda_2 y_2}.$$

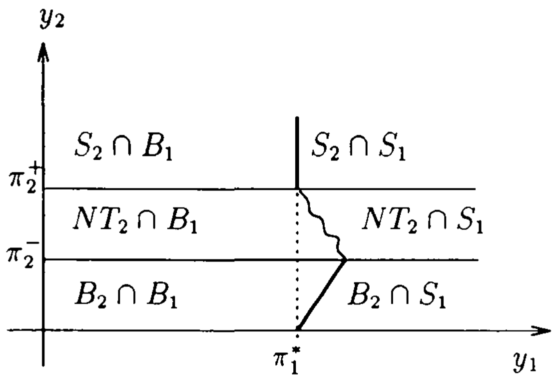


Figure 2: Boundaries of the transaction regions in the case of no transaction cost for the first risky asset.

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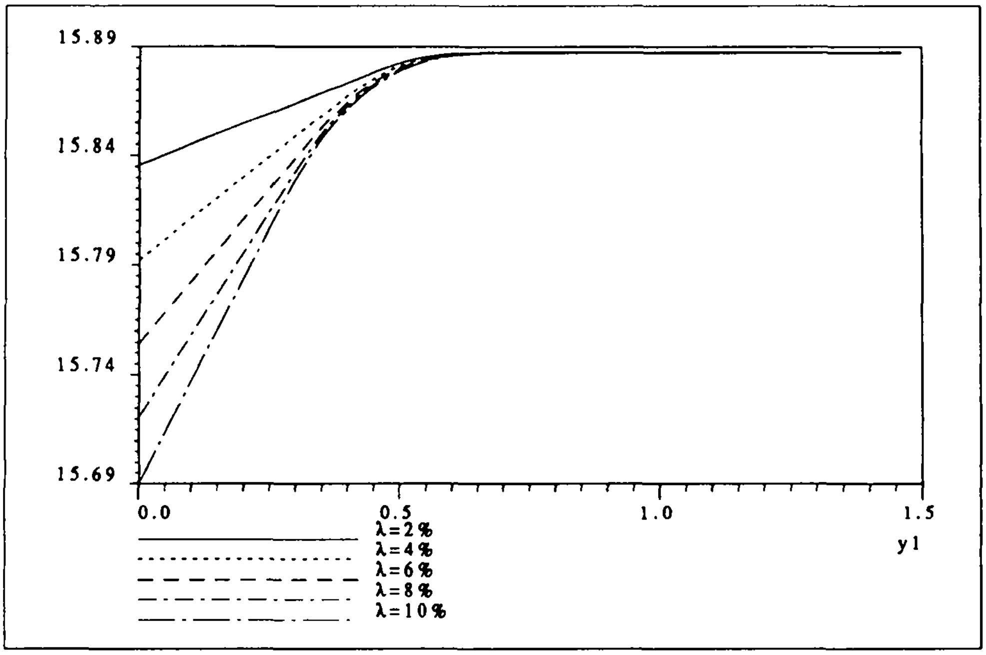


Figure 3: Value function $W$ for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\mu = 0$.

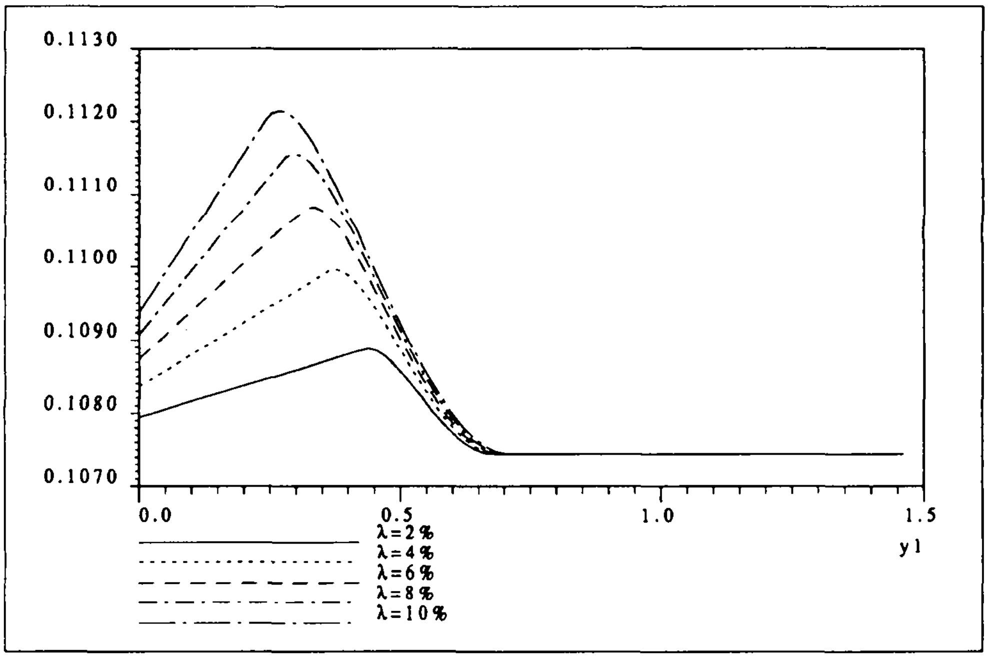


Figure 4: Optimal consumption $C$ for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\mu = 0$.

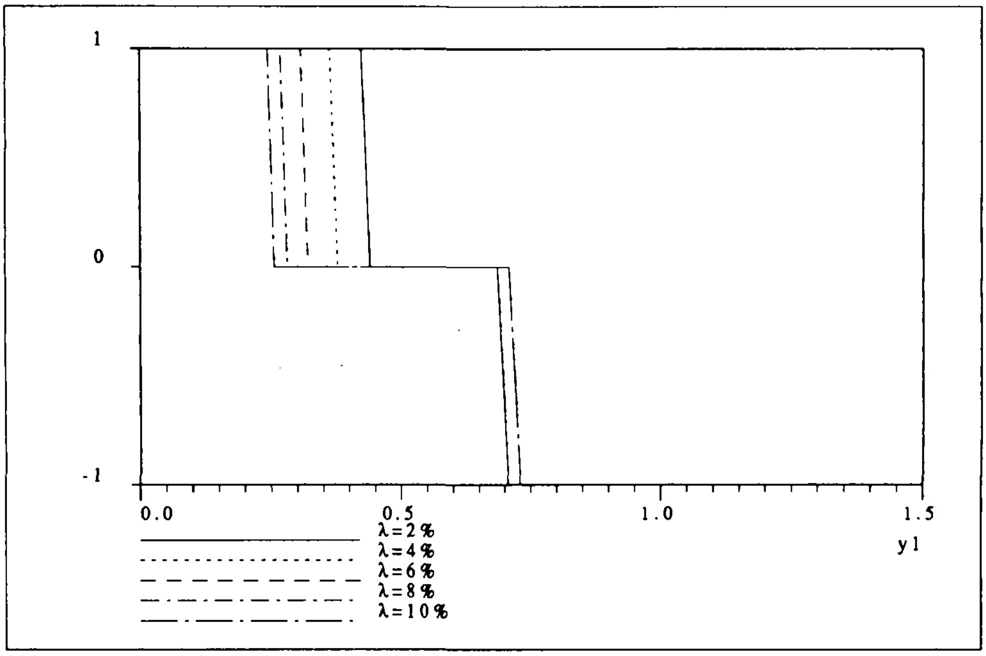


Figure 5: Graph of $1_{B_1} - 1_{S_1}$ for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\mu = 0$.

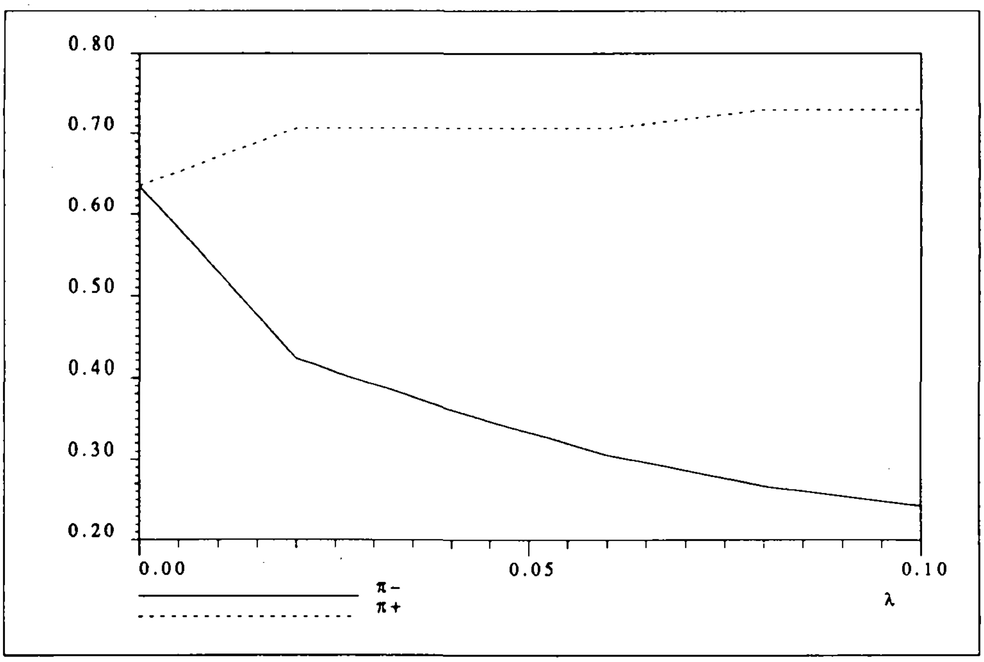


Figure 6: Graph of $\pi^+$ and $\pi^-$ for $n = 1$, $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha_1 = 11\%$, $\sigma_1 = 30\%$ and $\mu = 0$.

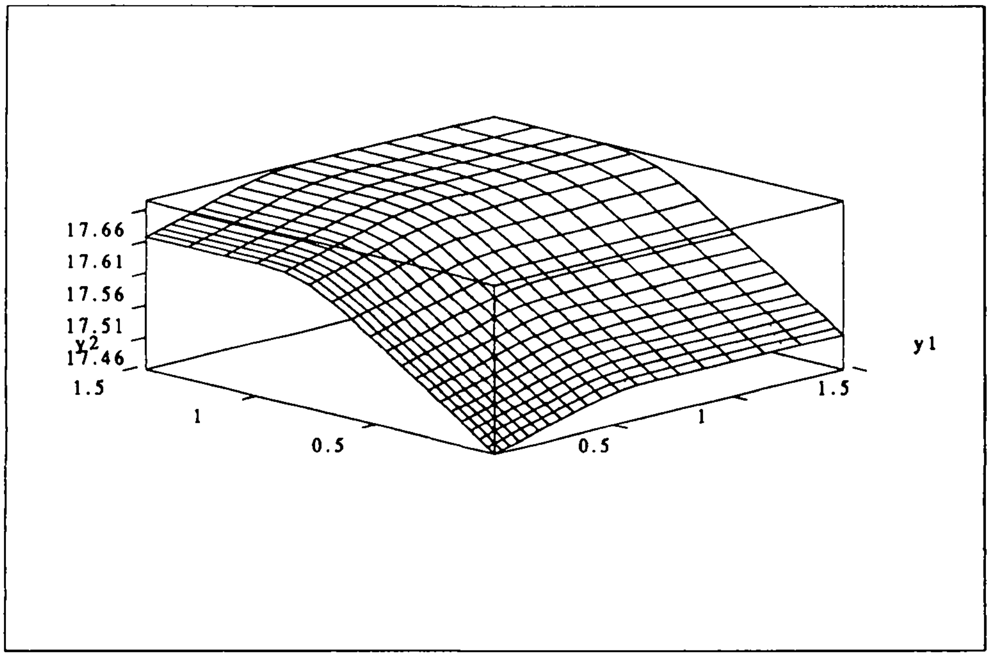


Figure 7: Value function $W$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

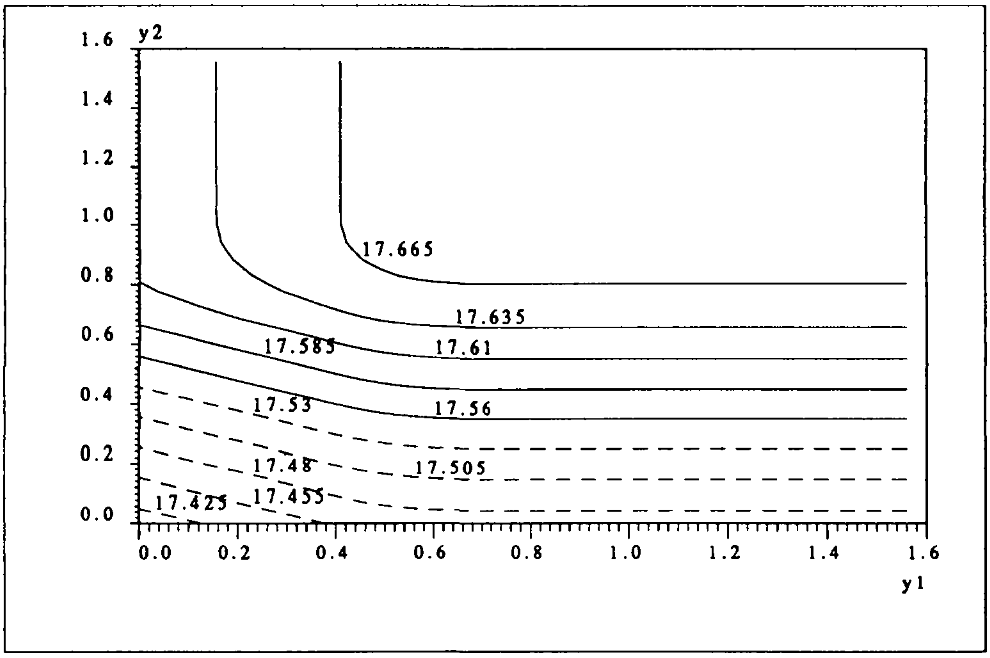


Figure 8: Value function $W$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

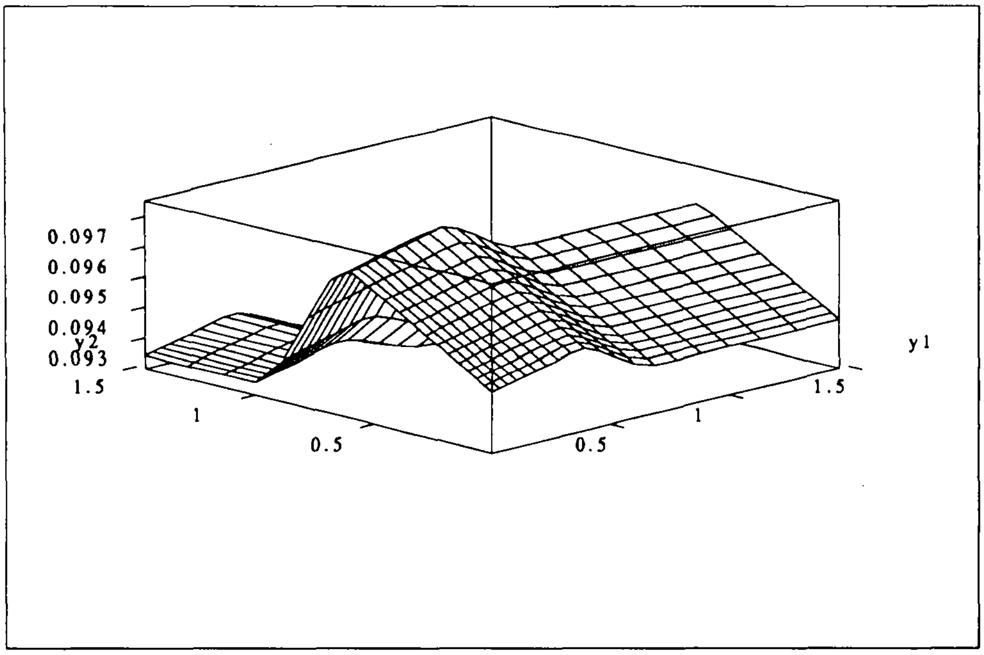


Figure 9: Optimal consumption $C$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

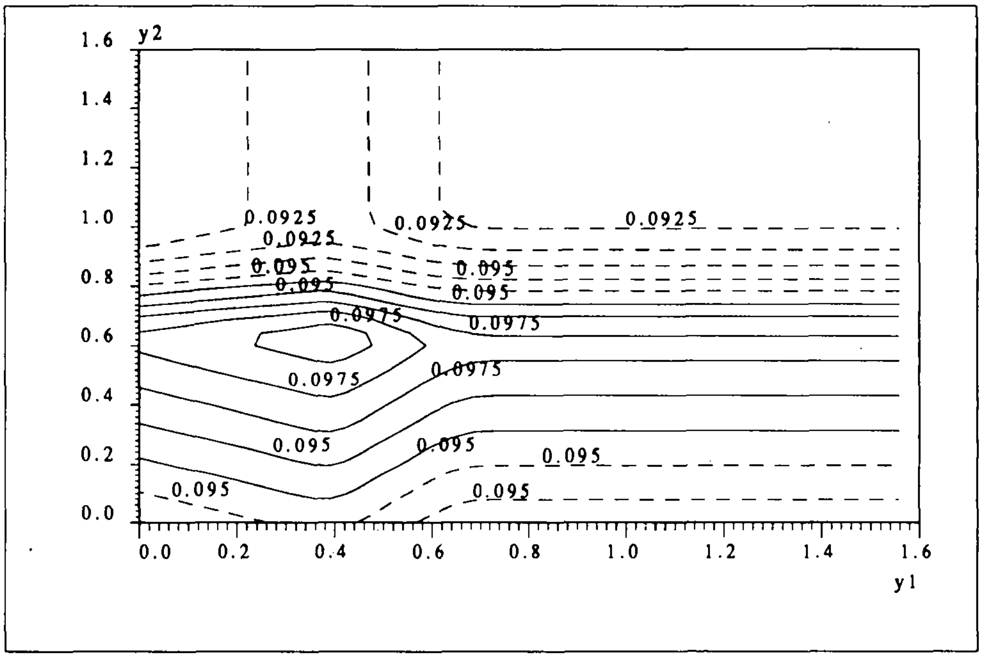


Figure 10: Optimal consumption $C$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

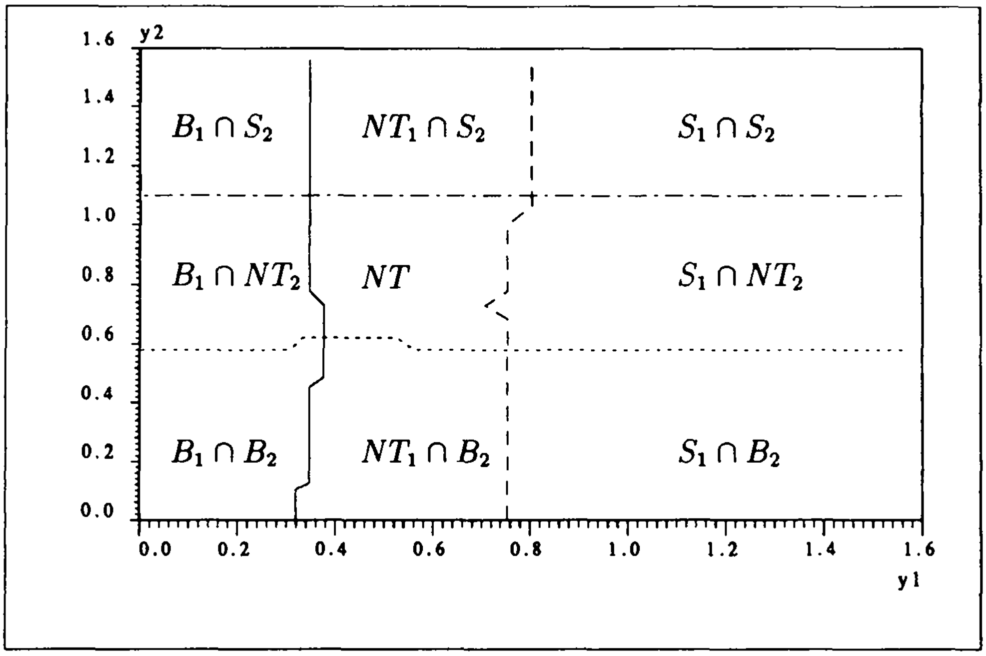


Figure 11: Boundaries of the regions $B_i$, $S_i$ and $NT_i$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

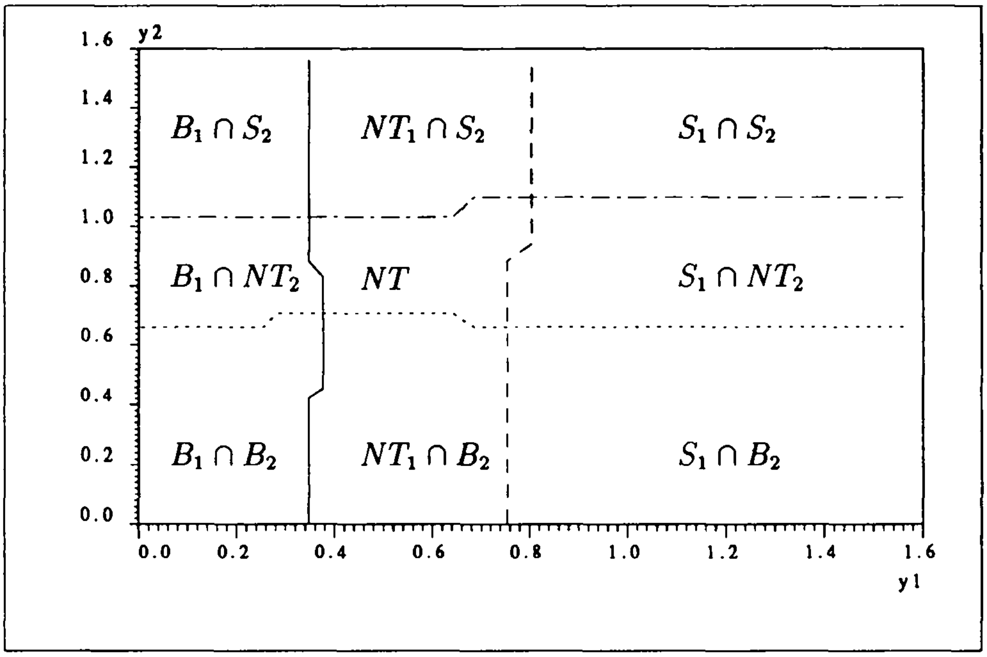


Figure 12: Boundaries of the regions $B_i$, $S_i$ and $NT_i$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 2\%)$ and $\mu = (0, 0)$.

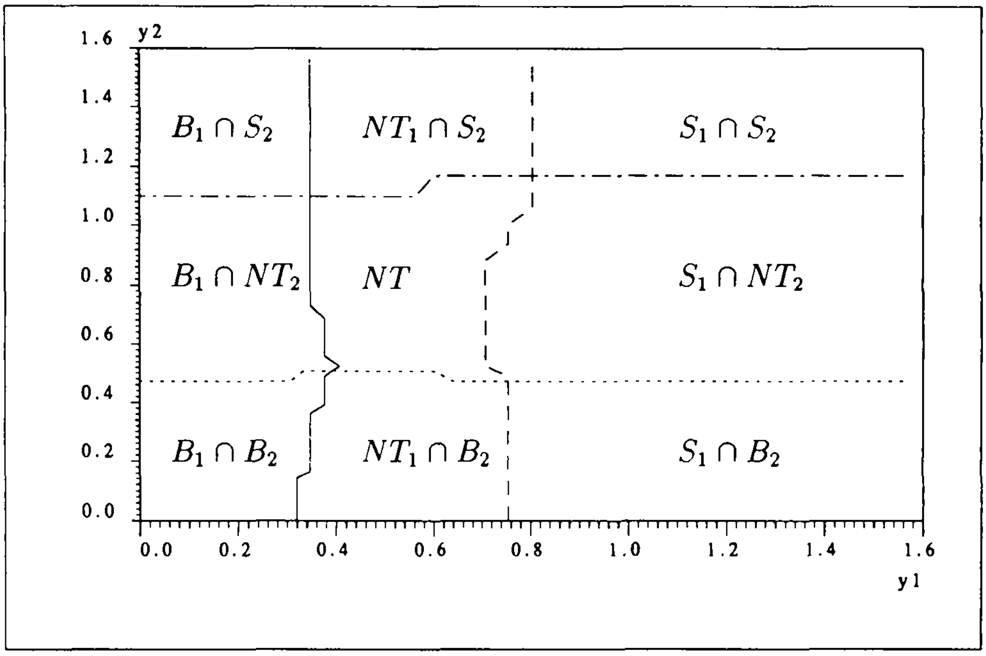


Figure 13: Boundaries of the regions $B_i$, $S_i$ and $NT_i$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 15\%)$, $\sigma = (30\%, 35\%)$, $\lambda = (2\%, 10\%)$ and $\mu = (0, 0)$.

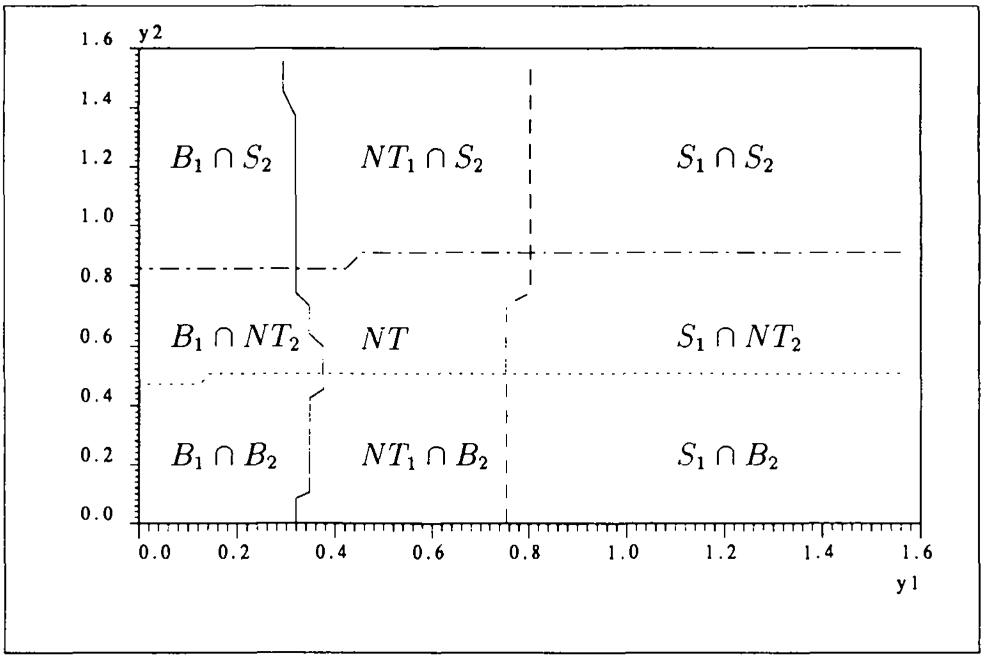


Figure 14: Boundaries of the regions $B_i$, $S_i$ and $NT_i$ for $\gamma = 0.3$, $\delta = 10\%$, $r = 7\%$, $\alpha = (11\%, 20\%)$, $\sigma = (30\%, 50\%)$, $\lambda = (2\%, 5\%)$ and $\mu = (0, 0)$.

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ISSN 0249 - 6399

*RR.1926*